ELSEVIER

Topology and its Applications 104 (2000) 255–291

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Conjugate unstable manifolds and their underlying geometrized Markov partitions

Gioia M. Vago [1]

Université de Bourgogne, U.F.R. des Sciences et Techniques, Département de Mathématiques, Laboratoire de Topologie U.M.R. 5584 du C.N.R.S., B.P. 47 870, 21078 Dijon Cedex, France

Received 30 January 1998; received in revised form 4 June 1998

Abstract

Conjugate unstable manifolds of saturated hyperbolic sets of Smale diffeomorphisms are characterized in terms of the combinatorics of their geometrized Markov partitions. As a consequence, the relationship between the local and the global point of view is also made explicit.

Keywords: Geometrized Markov partitions; Invariant manifolds; Smale diffeomorphisms

AMS classification: Primary 54H20, Secondary 58F15; 54F15

1. Introduction

Diffeomorphisms of compact surfaces satisfying Axiom A and strong transversality are briefly called *Smale diffeomorphisms*. We are interested here in the connections between topology and dynamics on the unstable manifold of compact invariant hyperbolic sets yielded by diffeomorphisms of this type.

Our approach being global, we will assume that the hyperbolic sets K we are dealing with are *saturated*: if two points belong to K, then the intersection of the stable manifold of one point with the unstable manifold of the other is contained in K, too. It is explained in [3, Section 2.3], how this notion generalizes the concept of *basic piece* appearing in Smale's classical spectral decomposition theorem (see [8]).

The relationship between topology and dynamics in this context is now completely exploited. First, in [3, Section 4.2] the authors construct a canonical (unique up to conjugacy) invariant neighborhood of a saturated set K, which they call *domain of* K

[1] Email: vago@u-bourgogne.fr.

0166-8641/00/$ – see front matter
PII: S0166-8641(99)00017-6

and denote by $\Delta(K)$. Next, they show that the dynamics on such domains is entirely determined by the germ of the diffeomorphism along K (see [3, Section 6.5]). Last, the assembling idea is that dynamics on domains can be characterized up to conjugacy via the so-called *geometrized Markov partitions*, which include and complete the combinatorial information contained in the classical Markov partitions (see [3, Section 6.2]).

Boundary leaves are the main tool in the construction of these partitions. In fact, the unstable manifold of a saturated set is a non-compact lamination (for a complete survey on both the existence and the structure of the invariant manifolds, see [5]). Its local structure in the surface is, at any point, a product $F \times [0,1]$ where F is a closed subset of $[0,1]$. A special role is therefore assigned to leaves which are accumulated by one side at most. In their local expression $E \times [0,1]$, $E \subset F$, the points of E are isolated (in F) either from the left or from the right. Leaves and points presenting such a behavior will be called *u-boundaries*, and *double u-boundaries* if the corresponding leaf is isolated from both sides. Analogously, we define *s-boundaries* and *double s-boundaries* for the stable lamination. Remark that if K has no s-boundaries, K is a hyperbolic attractor, while if K has no u-boundaries, K is a hyperbolic repeller.

The result constituting our starting point is a theorem by Bonatti and Langevin asserting that the simple topological knowledge of the texture woven by the invariant manifolds of K determines the dynamics (up to iteration) on the connected components of the domain of K. More precisely:

Theorem* [3, Theorem 7.0.6]. *Let f and g be two Smale diffeomorphisms and K and L two hyperbolic saturated sets without double boundaries, whose domains $\Delta(K)$ and $\Delta(L)$ are connected. Assume that there exists a homeomorphism $h: W^u(K) \cup W^s(K) \to W^u(L) \cup W^s(L)$ such that for all x in K we have $h(W^s(x)) = W^s(h(x))$ and $h(W^u(x)) = W^u(h(x))$.*

Then, there exist p and q in $\mathbb{N}$ such that $f^p|_{\Delta(K)}$ is conjugate to $g^q|_{\Delta(L)}$.

In the case of double boundaries, a conjecture is stated in [3, Section 7.4]. It essentially coincides with Theorem* but takes care of the fact that the connected components of $\Delta(K)$ minus the double boundaries can behave independently from the dynamical point of view.

Our discussion is motivated by the following question: *what is it left from this theory when we start from the simple topological knowledge of only the unstable manifold?*

The main difficulty arises from the loss of the transversal structure which, in the case dealt with by Theorem*, was given by the simultaneous presence of both the invariant manifolds.

This is the reason why, first of all, we cannot expect to have the same kind of results: in the unstable case, the “homeomorphic” level is clearly distinct from the “conjugacy” point of view and there is no way to make them equivalent in the general case. Consider for instance the two dynamical systems represented in Fig. 1 and obtained by the classical procedure of squeezing, stretching and bending originally used to describe Smale’s classical horseshoe map.

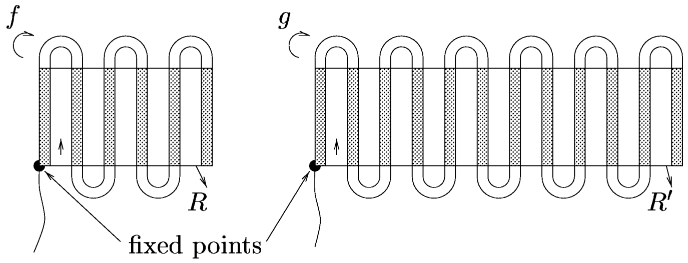

Fig. 1. Two homeomorphic unstable manifolds which cannot be conjugate.

Using Watkins's classification theorem for Knaster continua (see [9], but also [2] and [1]) we have that the corresponding unstable manifolds are homeomorphic. On the other hand, consider the topological entropies h and $\bar{h}$ of the two systems. It is: $h = \log 6$, and $\bar{h} = \log 12$. The quotient of h and $\bar{h}$ being irrational, no iterate of the first diffeomorphism can be conjugate to any iterate of the second.

We choose here to treat the "conjugacy problem". Given two unstable manifolds of saturated sets containing neither hyperbolic attractors, nor hyperbolic repellers, we establish a necessary and sufficient combinatorial condition in order for them to be conjugate (Theorem B below).

Remark that our assumption excluding attractors and repellers is not restrictive. In fact, in the case of hyperbolic repellers, the stable manifold is contained in the set K itself, hence the conjugacy problem is essentially solved by Theorem* (see also [3, Theorem 3.3.4]). As for hyperbolic attractors, a classification theorem already exists: we refer the reader to [10] for a comparison between Williams's approach and ours.

An important notion we will strongly make use of is that of the *unstable combinatorial type of a Markov partition* (see Section 2.2), which translates the information about the "sense" of the intersection of an image rectangle with a given one. For instance, the unstable combinatorial type of the dynamics in Fig. 1 just reveals that the image rectangles $f(R)$ and $g(R')$, respectively cut R and R' alternatively in the positive and negative direction six times for f and twelve for g.

A first result is stated by the following proposition which will turn out to be a corollary of Theorem B.

Proposition A. *Two unstable manifolds of saturated sets admitting Markov partitions with the same unstable combinatorial type are conjugate.*

As an easy example, the three dynamics represented in Fig. 2 satisfy the assumption. Their unstable combinatorial type consists in that the image of the rectangle R intersects R itself alternatively in the positive and negative direction four times. Thus, by Proposition A, their corresponding unstable manifolds are conjugate.

Let us briefly compare them (see next sections for rigorous definitions).

Being a leaf (of an unstable manifold) containing a *free separatrix* is a topological property (see Section 4.2). Hence, the leaves F_{x_1}, F_{x_2} and F_{x_3}, passing through x_1, x_2

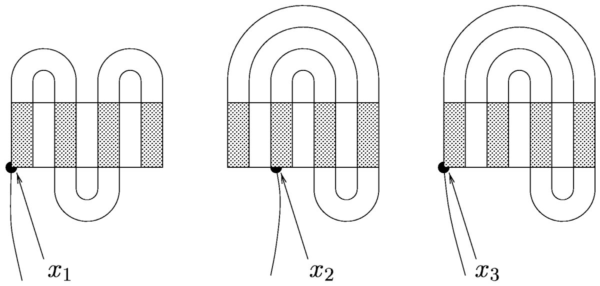

Fig. 2. Some one-rectangle dynamics yielding conjugate unstable manifolds.

and x_3, respectively, must be associated by any homeomorphism (that is, not necessarily a conjugacy) between the corresponding systems. Let h be the conjugacy between the unstable manifolds of systems 1 and 2, given by Proposition A. By the above remark, it must be $h(F_{x_1}) = F_{x_2}$. Now, the leaf F_{x_1} is u-boundary, i.e., isolated from one side, while the leaf F_{x_2} is not. Therefore, Examples 1 and 2 show that, for a leaf, the property of *being u-boundary is not preserved under homeomorphism*.

A direct consequence is that even by imposing to a homeomorphism the respect of the dynamics, there is *no way to extend its definition* (*as a plain homeomorphism*) *on an open neighborhood of the unstable manifold*. Moreover, Examples 1 and 3 show that the same remark can hold even when u-boundaries are preserved.

Unluckily we do not know the answer to the following

Question 1. Consider two conjugate unstable manifolds of hyperbolic saturated systems (K, f) and (L, g). Do there exist two geometrized Markov partitions $\{R_i\}_{i=1}^N$ for (K, f) and $\{Q_i\}_{i=1}^N$ for (L, g) admitting the same unstable combinatorial type?

In order to make the condition on the unstable combinatorial type necessary and sufficient, we will weaken the combinatorial requests on the Markov partitions.

The first expedient consists in *regrouping* the rectangles of a Markov partition into "packages" whose dynamical regrouped behavior enables us to define their *regrouped unstable combinatorial type*.

As an example of this regrouping procedure, consider the two systems represented in Fig. 3, respectively described by the Markov partitions $\{R_1, R_2\}$ and $\{Q_1, Q_2, Q_3, Q_4\}$.

The two packages $\{Q_1, Q_4\}$ and $\{Q_2, Q_3\}$ behave under g like R_1 and R_2, respectively do under f. In fact, $f(R_1)$ intersects R_1 and R_2 in the order, according to their positive orientation; the same is true for $g(Q_1 \cup Q_4)$ with respect to the oriented $Q_1 \cup Q_4$ and $Q_2 \cup Q_3$. Again, $g(Q_2 \cup Q_3)$ cuts, in the order, both $Q_2 \cup Q_3$ and $Q_1 \cup Q_4$ according to the negative orientation, as well as $f(R_2)$ does, with respect to the oriented R_2 and R_1. In other words, the regrouped unstable combinatorial type of the regrouped Markov partition $\{\{Q_1, Q_4\}, \{Q_2, Q_3\}\}$ is the same as the one corresponding to the Markov partition $\{R_1, R_2\}$ of the first system.

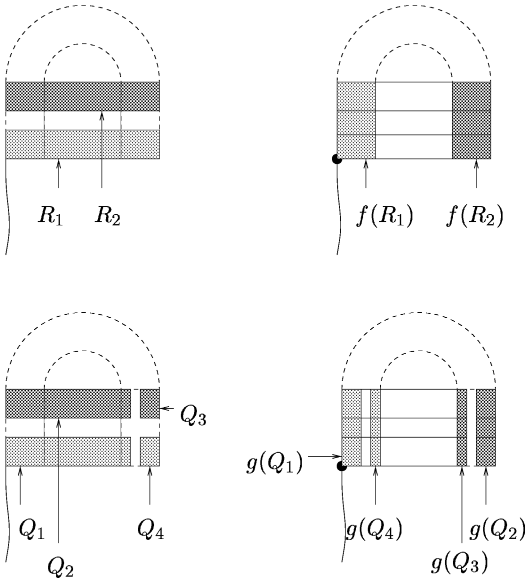

Fig. 3. Two Markov partitions with the same unstable combinatorial type up to regrouping.

However, the two resulting unstable manifolds cannot be conjugate, even up to iteration: for any $p \in \mathbb{N}$ and $q \in \mathbb{N}$, f^p and g^q do not admit the same number of fixed points.

The obstacle is made explicit by the following combinatorial fact. We can associate to the points of the hyperbolic set of the second system their itinerary with respect to packages (instead of the usual one by rectangles). This coding procedure by packages is not one-to-one, while the corresponding itinerary function (by rectangles) for the first system is.

In order for packages to supply an injective itinerary function, we will need a further assumption on the underlying regrouping structure, which can be read on the incidence matrix of the non-regrouped Markov partition. Since it can be checked on the non-existence of cycles in an oriented graph (see Section 2.4), the condition will be called *no double cycles*.

We will prove:

Theorem B. *Let $W^u(K)$ and $W^u(L)$ be the unstable manifolds of the saturated systems (K, f) and (L, g), respectively. Assume that K and L contain neither hyperbolic attractors, nor hyperbolic repellers. The following statements are equivalent*:

(1) *there exists a homeomorphism h from $W^u(K)$ onto $W^u(L)$ conjugating $f|_{W^u(K)}$ to $g|_{W^u(L)}$*;

(2) *for any generating Markov partition $\{R_i\}_{i=1}^N$ for (K, f) whose unstable combinatorial type is σ^u, there exists a generating Markov partition $\{Q_p\}_{p=1}^P$ for (L, g) and a regrouping structure $\{A_i\}_{i=1}^N$ for $\{1, \dots, P\}$ such that*:
- *the regrouped unstable combinatorial type τ^u of $\{\{Q_p\}_{p\in A_i}\}_{i=1}^N$ equals σ^u*;
- *the regrouping structure has no double cycles.*

Assuming (1), we will consider the restriction of h to the intersection of $W^u(K)$ with the rectangles of $\{R_i\}_{i=1}^N$, a Markov partition for (K, f). Topologically, such intersections are the product of special meager sets with an interval, that is, they are *matchboxes* (Section 3.1). The same is true for their images, but, because of the possible transversal rearrangement of the meager sets in the surface, we can only deduce that for all i, $h(W^u(K) \cap R_i)$ will be the trace of $W^u(L)$ on a finite number of rectangles $\{Q_p\}_{p\in A_i}$. The key of the proof is that they can be chosen as to constitute a Markov partition for (L, g).

Another delicate step consists in understanding the condition "no double cycles" discussed above in terms of the dynamics (Lemma 2.9), after which the proof is straightforward.

On the other hand, assume (2) holds. We will define the conjugacy h step by step. As for K, we will make use of the itinerary by packages. Since the unstable combinatorial types are the same up to regrouping, such a conjugacy is order preserving and makes a correspondence between the free separatrices of the two systems through their origin. After choosing a conjugacy for each orbit of free separatrices, we can complete the definition of h essentially via some transversal invariant foliations.

Proposition A is now an immediate corollary of Theorem B. First remark that there is no loss of generality in considering that the Markov partitions appearing in the assumption of Proposition A are generating. In this case, such an assumption coincides with hypothesis (2) in Theorem B when the regrouping structure is trivial, that is, each package is constituted by one and only one rectangle. The regrouped unstable combinatorial type is then the unstable combinatorial type itself. The "no double cycles" condition is redundant: packages supply an injective itinerary function since rectangles do.

With the help of Theorem B, we can characterize the relationship existing between the local and the global point of view. We prove that two unstable manifolds are globally conjugate if and only if they are locally homeomorphic via a homeomorphism which is a conjugacy when restricted to the hyperbolic sets.

Corollary C. *Under the same assumptions as in Theorem B, the following statements are equivalent*:

(1) *there exists a homeomorphism h from $W^u(K)$ onto $W^u(L)$ conjugating $f|_{W^u(K)}$ to $g|_{W^u(L)}$*;

(2) *there exists a conjugacy $\tilde{h}$ between the two hyperbolic sets K and L which can be locally extended to a homeomorphism $\hat{h}$ defined from the local unstable manifold $W^u_{loc}(K)$ onto the local unstable manifold $W^u_{loc}(L)$.*

In its proof we encounter some of the techniques already introduced to establish Theorem B. By looking at them, we can draw a parallel between the two assumptions in statement (2) of Corollary C and the two in statement (2) of Theorem B. The hypothesis "there exists a conjugacy $\tilde{h}$" takes the place of: "the regrouping structure has no double cycles", while the assumption on the local extension is a suitable substitute to the equality of the unstable combinatorial types.

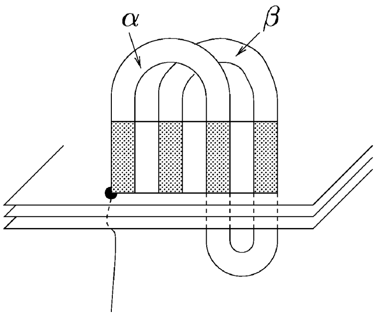

Fig. 4. A geometrized Markov partition of infinite genus.

Finally, we can ask ourselves about the relationship between the intrinsic and extrinsic topology of the unstable laminations.

According to [6] and [3], we say that a geometrized Markov partition is *realizable* if its nonwandering set corresponds to the saturated set of a dynamics living on a compact surface (thus of finite genus). Following Bonatti and Jeandenans, we can hence define the *genus of a geometrized Markov partition* as the minimal genus of the hosting surfaces if the partition is realizable, equal to the infinite if not. In [6] and [3, Chapter 8], it is shown that the possibility of having a real model of a geometrized Markov partition can be decided by a finite algorithm. The three examples in Fig. 2 turn out to be realizable (on the sphere, see references), while the one described in Fig. 4 is not: the crossing of the ribbons α and β gives a new contribution to the minimal genus at each iterate (see references).

Anyway, it is possible to embed this dynamics $(\widetilde{K}, \tilde{f})$ in a surface $\widetilde{S}$ of infinite genus contained in $\mathbb{R}^3$ (the ribbons α and β and their iterates just need one more dimension to cohabit). Moreover, the natural choice of such an embedding endows $\widetilde{S}$ with a transversal invariant stable foliation which is the restriction to $\widetilde{S}$ of a two-dimensional foliation defined in a neighborhood (in $\mathbb{R}^3$) of $W^u(\widetilde{K})$. By the same arguments of our proof (Section 4), there exists a conjugacy between the unstable manifold of this system and the unstable manifold of any of the dynamics shown in Fig. 2.

The comparison between these examples shows that *the possibility of having a real model for a geometrized Markov partition is an extrinsic property*: it depends on the embedding in the surface. Nevertheless, we can ask ourselves if, when finite, the genus is invariant under conjugacy on the unstable manifolds:

Question 2. Are there examples of two realizable geometrized Markov partitions yielding conjugate unstable manifolds whose genera are different? Is the answer the same when we replace “conjugate” by “homeomorphic”?

In Section 2 we introduce the tools which are necessary to state precisely our theorems. Next, Sections 3 and 4 are completely devoted to the proof of Theorem B. In particular, Proposition A is recalled in Section 4. Corollary C is dealt with in Section 5.

2. Understanding the combinatorial conditions

2.1. Geometrized Markov partitions

Let us consider a Smale orientation preserving diffeomorphism f on an oriented compact surface S, and a hyperbolic saturated set K. According to [3, Chapters 5 and 6], the dynamics f on an invariant canonical neighborhood $\Delta(K)$ of K can be completely described (up to conjugacy) through a special combinatorial action Φ (see Definition 2.1 and Theorem 2.2 below).

Let $J_1 = J_2 = [0, 1]$ and $h : J_1 \times J_2 \to M$ be a homeomorphism onto its image. We will call $h(J_1 \times J_2) = R$ *rectangle* if it is trivially laminated by the invariant manifolds (i.e., for every $t \in J_1$, $h(J_1 \times \{t\})$ is either disjoint or included in the stable manifold $W^s(K)$ and, symmetrically, for every $t \in J_2$, $h(\{t\} \times J_2)$ is either disjoint or included in the unstable manifold $W^u(K)$) and if for $t = 0$ and $t = 1$ such inclusions hold. We denote by $\partial^s R$ the *stable boundary* of R, i.e., $h(J_1 \times \{0\}) \cup h(J_1 \times \{1\})$, as well as $\partial^u R$ will stand for its *unstable boundary* $h(\{0\} \times J_2) \cup h(\{1\} \times J_2)$.

In order to describe all situations, we are in the need to consider also *degenerate rectangles*, that is, rectangles for which $J_1 = \{0\}$ and/or $J_2 = \{0\}$. We will still call them *rectangles*.

We define a *horizontal subrectangle* of R as a rectangle $H \subset R$ such that $\partial^u H \subset \partial^u R$, horizontally crossing R all along (there exist t_1 and t_2 in J_2 such that $\partial^s H = h(J_1 \times \{t_1\}) \cup h(J_1 \times \{t_2\})$).

Analogously, a *vertical subrectangle* of R will be a rectangle $V \subset R$ such that $\partial^s V \subset \partial^s R$, vertically crossing R all along (there exist t_3 and t_4 in J_1 such that $\partial^u V = h(\{t_3\} \times J_2) \cup h(\{t_4\} \times J_2)$).

A finite number $\{R_i\}_{i=1}^N$ of rectangles covering K is said to be a *good Markov partition* if the following conditions are satisfied:

- for all $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, N\}$ such that $i \neq j$ the rectangles R_i and R_j are disjoint, that is, their distance is bounded away from zero;
- for every couple of indexes (i, j), each connected component of $R_i \cap f(R_j)$ is a vertical subrectangle V_i^k of R_i;
- for every vertical subrectangle V_i^k of R_i obtained as above, the corresponding preimage $f^{-1}(V_i^k)$ is a horizontal subrectangle H_j^l of R_j;
- for every $i = 1, \ldots, N$ each connected component of $\partial^u R_i$ is the unstable boundary of a vertical subrectangle obtained as above, and respectively, each connected component of $\partial^s R_i$ is the stable boundary of a horizontal subrectangle obtained as above;
- for every sequence $\{i_n\}_{n\in\mathbb{Z}}$, each connected component of $\bigcap_{n\in\mathbb{Z}} f^n(R_{i_n})$ contains at most one point belonging to K.

Hence, the definition of good Markov partition allows us to talk about horizontal and vertical subrectangles which are mapped into each other by f. An intuitive idea is given by Fig. 5.

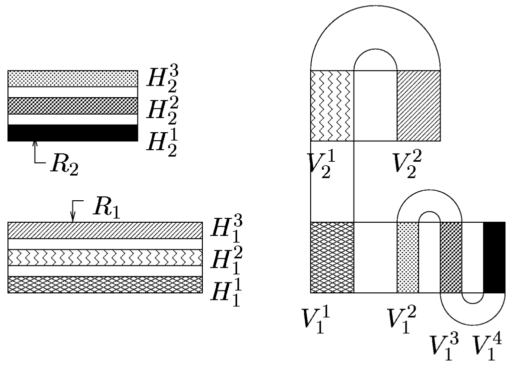

Fig. 5. Horizontal and vertical subrectangles of a Markov partition.

As pointed out before, for every couple of indexes (i, j), the connected components of $R_i \cap f(R_j)$ are vertical subrectangles V_i^k of R_i. They can also be seen as horizontal subrectangles of $f(R_j)$, so that their preimages are horizontal subrectangles H_j^l of R_j. In Fig. 5, the rectangle R_1 has three horizontal subrectangles and four vertical subrectangles, while R_2 has three horizontal subrectangles but two vertical subrectangles. It is $f(H_1^1) = V_1^1$, $f(H_1^2) = V_2^1$, $f(H_1^3) = V_2^2$ and so on.

A good Markov partition is called *generating* if for every sequence $\{i_n\}_{n\in\mathbb{Z}}$ the total intersection $\bigcap_{n\in\mathbb{Z}} f^n(R_{i_n})$ contains at most one point.

Remark that the orientation of M induces an orientation ω_i on the rectangles R_i's, thus on their vertical lines (for which we keep the same notation ω_i), after having chosen one for the horizontal. In the degenerate case, degenerate directions can be "morally" oriented. We are therefore allowed to speak about top, bottom, left and right, in restriction to rectangles.

To fix our notations, let us consider

(1) a positive integer N, representing the number of rectangles in the partition,
(2) two sets of positive integers $\{h_i\}_{i=1}^N$ and $\{v_i\}_{i=1}^N$ (which denote the number of the horizontal and vertical subrectangles of R_i) such that $\sum_{i=1}^N h_i = \sum_{i=1}^N v_i$,
(3) N collections of sets of the type $\{H_i^j\}_{j=1}^{h_i}$ for $i = 1, \ldots, N$ and N collections of sets of the type $\{V_i^k\}_{k=1}^{v_i}$ for $i = 1, \ldots, N$, namely, the horizontal and vertical subrectangles of the R_i's themselves with the convention that horizontal subrectangles are listed from bottom to top, while vertical subrectangles are numbered from left to right.

Definition 2.1. An abstract geometrical type is given by $(N, \{h_i\}_{i=1}^N, \{v_i\}_{i=1}^N)$ and by a map Φ defined as below, which is a signed bijection:

$$\Phi : \bigcup_{i=1}^N \{i\} \times \{1, \ldots, h_i\} \to \bigcup_{k=1}^N \{k\} \times \{1, \ldots, v_k\} \times \{+, -\}$$
$$(i, j) \to (k, l, \varepsilon).$$

The geometrical type of a Markov partition (or, equivalently, a geometrized Markov partition) is the abstract geometrical type Φ associated to the dynamical action f such that $f(H_i^j) = V_k^l$ and $f(\omega_i|_{H_i^j}) = \varepsilon \cdot \omega_k|_{V_k^l}$ if and only if $\Phi(i, j) = (k, l, \varepsilon)$.

For instance, for the one-rectangle Markov partition of Example 1 in Fig. 2, it is $N = 1$, $h_1 = v_1 = 4$ and $\Phi(1, 1) = (1, 1, +)$, $\Phi(1, 2) = (1, 2, -)$, $\Phi(1, 3) = (1, 3, +)$ and $\Phi(1, 4) = (1, 4, -)$.

It is shown in [3, Chapters 5 and 6], that Φ contains the optimal data not to lose any dynamical information on the canonical neighborhood $\Delta(K)$, as stated below.

Theorem 2.2. *Let f and g be two Smale diffeomorphisms defined on compact surfaces, and K and L two hyperbolic saturated sets of f and g respectively, which contain neither hyperbolic attractors, nor hyperbolic repellers. Then f and g are conjugate on the corresponding $\Delta(K)$ and $\Delta(L)$ if and only if (K, f) and (L, g) admit Markov partitions of the same geometrical type.*

It will be useful to remark that the rectangles of a Markov partition are provided with an invariant stable foliation in a neighborhood of them, as stated by Proposition 6.3.1 in [3]:

Proposition 2.3. *Let K be a saturated hyperbolic set of a Smale diffeomorphism f. Then, there exists an invariant neighborhood U of K such that*

(1) *there exists a stable foliation F^s defined on U, invariant for f, transversal to $W^u(K)$, containing* (*the restrictions on U of*) *the leaves of the stable lamination $W^s(K)$ as its own leaves,*

(2) *any geometrized Markov partition is covered by U.*

2.2. Unstable combinatorial types

If the position of a vertical subrectangle with respect to the other vertical subrectangles of the same rectangle is of no interest to us, the information given by the geometrical type Φ is redundant. This is the reason why we introduce the *unstable combinatorial type* which is obtained from a geometrical type by forgetting about the second element of $\Phi(i, j)$.

Definition 2.4. An abstract unstable combinatorial type is given by $(N, \{h_i\}_{i=1}^N)$ and by an application

$$\sigma^u : \bigcup_{i=1}^N \{i\} \times \{1, \ldots, h_i\} \to \{1, \ldots, N\} \times \{+, -\}$$
$$(i, j) \to (k, \varepsilon).$$

The unstable combinatorial type of an oriented Markov partition is the abstract unstable combinatorial type defined as $\sigma^u(i, j) = (k, \varepsilon)$ if and only if there exists $l \in \{1, \ldots, v_k\}$ such that $\Phi(i, j) = (k, l, \varepsilon)$.

For instance, for all the geometrized Markov partitions represented in Fig. 2, it is $\sigma^u(1,1) = (1,+)$, $\sigma^u(1,2) = (1,-)$, $\sigma^u(1,3) = (1,+)$ and $\sigma^u(1,4) = (1,-)$.

Remark that if we start from a generating Markov partition, the restrictions $\sigma^u|_{\{i\}\times\{1,\dots,h_i\}}$ are signed injections for all $i = 1,\dots,N$.

The idea lying behind this definition is that we have made a quotient of the elements appearing in a geometrized Markov partition by some invariant foliation. More precisely, let F^s be a foliation as the one in Proposition 2.3. For every $i = 1,\dots,N$, let F_i^s be the restriction of F^s to R_i denoted by $F_i^s = F^s|_{R_i}$. Define I_i as the space of the leaves of F_i^s, that is, $I_i = R_i/F_i^s$. Hence, each I_i can be trivially identified with, for instance, one of the connected components of $\partial^u R_i$, which is an oriented (maybe degenerate) closed interval. Thus, so is I_i. Still call ω_i its orientation.

Repeat now the same procedure for every $j = 1,\dots,h_i$ and $i = 1,\dots,N$, in order to define J_i^j, the space of the leaves of F_i^s in restriction to H_i^j, that is, $J_i^j = H_i^j/F_i^s$. For every $i = 1,\dots,N$, we have that for every $j = 1,\dots,h_i$ the space J_i^j is a maybe degenerate oriented closed subinterval of I_i.

With this background (see [3, Section 6.1]), f gives naturally rise to a one-dimensional *Markovian* function f^u on $J^u := \bigcup_{i=1}^N \{J_i^j\}_{j=1}^{h_i}$ onto $I^u := \bigcup_{i=1}^N I_i$, which is called the *unstable component of* f. The map σ^u corresponds to the Markovian action f^u such that $f^u(J_i^j) = I_k$ and $f(\omega_i|_{J_i^j}) = \varepsilon\cdot\omega_k|_{I_k}$ if and only if $\sigma^u(i,j) = (k,\varepsilon)$.

The following diagram clarifies the entire procedure.

$$\begin{tikzcd}
H_i^j \arrow[r, "f"] \arrow[d, "\text{Quotient by } F^s|_{R_i}"'] & V_k^l \arrow[d, "\text{Quotient by } F^s|_{R_k}"] & \rightsquigarrow & (i,j) \arrow[r, "\Phi"] \arrow[d] & (k,l,\varepsilon) \arrow[d] \\
J_i^j \arrow[r, "f^u"] & I_k & \rightsquigarrow & (i,j) \arrow[r, "\sigma^u"] & (k,\varepsilon)
\end{tikzcd}$$

In some sense, the quotient operation has erased the transversal information: at any scale, the mutual position of vertical subrectangles cannot be recovered (we can only know to which rectangle R_k a given vertical subrectangle $f(H_i^j)$ belongs, but it is impossible to recognize it among all the vertical subrectangles $\{V_k^l\}_{l=1}^{v_k}$ of R_k).

2.3. Regrouping operations

Let $\{R_i\}_{i=1}^N$ be a Markov partition and σ^u its unstable combinatorial type. Consider a partition of the indexes $\{1,\dots,N\}$ into M classes of the form $A_1 = \{1,\dots,a_1\}$, $A_2 = \{a_1+1,\dots,a_2\},\dots,A_M = \{a_{M-1}+1,\dots,a_M\}$ (where $0 < a_l < a_{l+1} \leqslant N$ for all $l = 1,\dots,M$). This partition $\{A_l\}_{l=1}^M$ is called *regrouping structure* if, up to renaming the rectangles of the Markov partition, for the corresponding packages $\{R_1,\dots,R_{a_1}\}$, $\{R_{a_1+1},\dots,R_{a_2}\},\dots,\{R_{a_{M-1}+1},\dots,R_{a_M}\}$, we have that:

(1) the $a_l - a_{l-1}$ rectangles $\{R_i\}_{i\in A_l}$ of the same package have the same number h_{a_l} of horizontal subrectangles;

(2) there exists an application

$$\tau^u : \bigcup_{l=1}^{M} \{l\} \times \{1, \ldots, h_{a_l}\} \to \{1, \ldots, M\} \times \{+, -\}$$

$$(l, j) \to (m, \varepsilon) = (m(l, j), \varepsilon(l, j))$$

such that: for any fixed $l \in \{1, \ldots, M\}$ and for any fixed $j \in \{1, \ldots, h_{a_l}\}$, the image of the jth horizontal subrectangle of any rectangle R_i in the package $\{R_i\}_{i \in A_l}$ is a vertical subrectangle of a rectangle R_k in the package $\{R_k\}_{k \in A_{m(l,j)}}$, while the image orientation is given by $\varepsilon(l, j) \cdot \omega_k$.

Definition 2.5. A Markov partition $\{R_i\}_{i=1}^N$ provided with a regrouping structure $\{A_l\}_{l=1}^M$ will be called regrouped Markov partition and denoted by $\{\{R_i\}_{i \in A_l}\}_{l=1}^M$. The map τ^u defined above will be called regrouped unstable combinatorial type.

Here is the relationship between the unstable combinatorial type σ^u of $\{R_i\}_{i=1}^N$ and the regrouped unstable combinatorial type τ^u of $\{\{R_i\}_{i \in A_l}\}_{l=1}^M$. Let $\mathcal{P}$ be the projection of $\{1, \ldots, N\}$ onto $\{1, \ldots, M\}$ defined as $\mathcal{P}(i) = l$ if $i \in A_l$. The properties defining a regrouped Markov partition guarantee: first, that for all $i \in A_l$, it is $h_i = h_{a_{\mathcal{P}(i)}} = h_{a_l}$; secondly, that for all $j = 1, \ldots, h_{a_l}$ there exist $m = m(\mathcal{P}(i), j)$ in $\{1, \ldots, M\}$ and $\varepsilon = \varepsilon(\mathcal{P}(i), j)$ in $\{+, -\}$, such that for all $i \in A_l$ it is $\sigma^u(i, j) = (k, \varepsilon)$, where $k \in A_m$. Hence, the following diagram commutes.

$$\begin{tikzcd}
(i, j) \arrow[r, "\sigma^u"] \arrow[d, "\mathcal{P} \times \mathcal{I}d_{\mathbb{N}}"'] & (k, \varepsilon) \arrow[d, "\mathcal{P} \times \mathcal{I}d_{\{+,-\}}"] \\
(\mathcal{P}(i), j) \arrow[r, "\tau^u"] & (\mathcal{P}(k), \varepsilon)
\end{tikzcd}$$

Remark that from the unstable combinatorial type σ^u it is possible to reconstruct the incidence matrix $D = (d_{i,j}) \in \mathcal{M}_N$ of the corresponding Markov partition $\{R_i\}_{i=1}^N$. It is: $d_{i,j} = card\{k \in \{1, \ldots, h_i\}$ such that $\sigma^u(i, k) = (j, \varepsilon),\ \varepsilon \in \{+, -\}\}$. These data can pass to the quotient by packages, too, and we can define a matrix containing the information about the incidence of packages.

Definition 2.6. The regrouped incidence matrix of a regrouped Markov partition $\{\{R_i\}_{i \in A_l}\}_{l=1}^M$ is a matrix $B = (b_{k,l}) \in \mathcal{M}_M$ for which $b_{k,l} = card\{j \in \{1, \ldots, h_{a_k}\}$ such that $\tau^u(k, j) = (l, \varepsilon)$, $\varepsilon \in \{+, -\}\}$.

Since τ^u is the regrouped unstable type of σ^u, the matrices D and B are linked together as follows.

Lemma 2.7. *Let $\{R_i\}_{i=1}^N$ be a generating Markov partition. Then the coefficients $d_{i,j}$ of the incidence matrix D and $b_{k,l}$ of the regrouped incidence matrix B belong to $\{0, 1\}$.*

Moreover, the block $D_{k,l}$ of the elements $d_{i,j}$ of D such that $i \in A_k$ and $j \in A_l$, is the zero block if and only if the corresponding element $b_{k,l}$ in B is the number zero. Otherwise, *if $b_{k,l}=1$, each row in $D_{k,l}$ is all composed by zeroes except for exactly one 1, whose position, depending of course on the regrouped partition, is not uniquely determined by B.*

Proof. The coefficient $d_{i,j}$ of the incidence matrix D coincides with the number of connected components of $f(R_i)\cap R_j$. The Markov partition being generating, such a number is either 0 or 1.

On the other hand, by checking the definition, the coefficient $b_{k,l}$ of the regrouped incidence matrix B is the number of connected components of $f(R_i)\cap(\bigcup_{p\in A_l}R_p)$, where R_i is any rectangle of the family $\{R_i\}_{i\in A_k}$. By definition of a regrouping structure, not only such a cardinality is independent of the choice of i in A_k, but we also have:

- $b_{k,l}=0$ if and only if $f(R_i)\cap(\bigcup_{p\in A_l}R_p)=\emptyset$ for all $i\in A_k$, if and only if $d_{i,p}=0$ for all $i\in A_k$ and $p\in A_l$, i.e., if and only if the block $D_{k,l}$ is the zero block;
- $b_{k,l}=1$ if and only if for all $i\in A_k$ there exists a unique $j\in A_l$ such that $f(R_i)\cap R_j$ is a vertical subrectangle of R_j, while $f(R_i)\cap R_p=\emptyset$ for all $p\in A_l\setminus\{j\}$. For a fixed $i\in A_k$, this means that $d_{i,j}=1$ and $d_{i,p}=0$ for all $p\in A_l\setminus\{j\}$.

Besides, if only the regrouped partition is known, there is no way to determine such a j in A_l. $\square$

Another consequence of the fact that we are considering generating Markov partitions, is that there exists a classical coding procedure φ for the points of the hyperbolic set K (see [4]). It associates to $x\in K$ its *itinerary* $\varphi(x)$ by:

$$\begin{aligned}&\varphi: K\to\varphi(K)=\Sigma\subset\{1,\ldots,N\}^{\mathbb{Z}}\\&\qquad x\to(\ldots,x_0,x_1,\ldots),\quad \text{where } x_k=j \text{ if } f^{-k}(x)\in R_j,\ \text{for } k\in\mathbb{Z}\end{aligned}$$

and conjugates $f^{-1}|_K$ to the subshift of finite type $\sigma|_\Sigma$. The set Σ can also be defined as the set of the bi-infinite sequences $(\ldots,x_0,x_1,\ldots)$ such that $d_{x_ix_{i-1}}=1$.

On the other hand, we have that $B=(b_{k,l})\in\mathcal{M}_M(\{0,1\})$ and we can also associate to a point $x\in K$ its *regrouped itinerary* $\tilde\varphi(x)$ in the natural way:

$$\begin{aligned}&\tilde\varphi: K\to\tilde\varphi(K)=\widetilde\Sigma\subset\{1,\ldots,M\}^{\mathbb{Z}}\\&\qquad x\to(\ldots,y_0,y_1,\ldots),\quad \text{where } y_k=l \text{ if } f^{-k}(x)\in\{R_j\}_{j\in A_l},\ \text{for } k\in\mathbb{Z}.\end{aligned}$$

Here, the set $\widetilde\Sigma$ is the set of the bi-infinite sequences $(\ldots,y_0,y_1,\ldots)$ for which $b_{y_ky_{k-1}}=1$.

Let $\pi=\mathcal{P}^{\mathbb{Z}}$ be the projection from $\{1,\ldots,N\}^{\mathbb{Z}}$ onto $\{1,\ldots,M\}^{\mathbb{Z}}$ acting as $\mathcal{P}$ on each element: $\pi(\ldots,x_0,x_1,\ldots)=(\ldots,\mathcal{P}(x_0),\mathcal{P}(x_1),\ldots)$. The relationship between φ and $\tilde\varphi$ is then given by:

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\mathcal{I}d_K"'] & \Sigma \arrow[d, "\pi=\mathcal{P}^{\mathbb{Z}}"] \\ K \arrow[r, "\tilde\varphi"] & \widetilde\Sigma \end{tikzcd}$$

Remark that the projection π is onto. The problem is that, in general, π (or equivalently, $\tilde\varphi$) may not be one-to-one. Consider as an example the second system in Fig. 3 and the regrouped Markov partition $\{\{Q_1,Q_4\},\{Q_2,Q_3\}\}$: for instance, the points of K lying on the right unstable border of Q_2 have the same regrouped itinerary as the points of K lying on the left unstable border of Q_3. In order not to lose any information when applying the quotient map π, a further condition is therefore needed.

2.4. No double cycles

We give here a necessary and sufficient combinatorial condition in order to avoid ambiguities when considering packages of rectangles instead of the rectangles themselves.

Definition 2.8. Let D be the incidence matrix of the generating Markov partition $\{R_i\}_{i=1}^N$ which is provided with a regrouping structure $\{A_l\}_{l=1}^M$. Associate to D the oriented graph whose vertices are the couples (i,j) of indexes belonging to the same package $A_{\mathcal{P}(i)}$, and whose arrows $(i,j)\to(k,l)$ connect the couples for which $d_{k,i}=d_{l,j}=1$. We say that the regrouping structure admits no double cycles if there exists no cycle of the form $(i_0,j_0)\to\cdots\to(i_n,j_n)=(i_0,j_0)$ such that $i_k\neq j_k$ for all $k=1,\dots,n$.

Lemma 2.9. *Let $\{R_i\}_{i=1}^N$ be a generating Markov partition provided with a regrouping structure $\{A_l\}_{l=1}^M$. They are equivalent*:
(1) *the regrouping structure admits no double cycles*;
(2) *the projection $\pi:\Sigma\to\widetilde{\Sigma}$ is a bijection*;
(3) *the map $\tilde{\varphi}:K\to\widetilde{\Sigma}$ induces a conjugacy between $f^{-1}|_K$ and $\sigma|_{\widetilde{\Sigma}}$.*

The key-lemma for this equivalence is the following:

Lemma 2.10. *Let $(\dots,x_0,x_1,\dots)$ and $(\dots,y_0,y_1,\dots)$ be the itineraries in Σ of two points of K. If there exists $n\in\mathbb{Z}$ such that $x_{n+1}=y_{n+1}$ and $\mathcal{P}(x_n)=\mathcal{P}(y_n)$, then $x_n=y_n$.*

Proof. In the row $\{d_{x_{n+1},z}\}_{z\in A_{\mathcal{P}(x_n)}}$ of the block $D_{\mathcal{P}(x_{n+1}),\mathcal{P}(x_n)}$, which is also the row $\{d_{y_{n+1},z}\}_{z\in A_{\mathcal{P}(y_n)}}$ of the block $D_{\mathcal{P}(y_{n+1}),\mathcal{P}(y_n)}$, there exists only one element $d_{x_{n+1},x_n}=1$, by Lemma 2.7. □

Proof of Lemma 2.9. For the equivalence (1) ⇔ (2) we argue by contradiction.

(1) ⇒ (2) We have already mentioned that the projection π is onto. We show that (1) implies that π is one-to-one. Suppose there exist two itineraries $(\dots,x_0,x_1,\dots)$ and $(\dots,z_0,z_1,\dots)$ in Σ such that for all $k\in\mathbb{Z}$, $\mathcal{P}(x_k)=\mathcal{P}(z_k)=y_k$ and for which there exists $k_0\in\mathbb{Z}$ such that $x_{k_0}\neq z_{k_0}$. By Lemma 2.10, we have that $x_k\neq z_k$ for all $k\geqslant k_0$. Let $s\in\{1,\dots,M\}$ be an index appearing infinitely many times in the sequence $\{\mathcal{P}(x_k)\}_{k\geqslant k_0}$. Since the number of couples (i,j) in $A_s\times A_s$ is finite, there exist x_l, x_{l+m}, z_l and z_{l+m} (all in A_s) such that $x_{l+m}=x_l$ and $z_{l+m}=z_l$. Then the existence of the double cycle $(x_l,z_l)\to(x_{l+1},z_{l+1})\to\cdots\to(x_{l+m},z_{l+m})$ is contrary to our assumption.

(2) ⇒ (1) Suppose there exists a cycle of the form: $(i_0,j_0)\to\cdots\to(i_n,j_n)=(i_0,j_0)$ such that $i_k\neq j_k$ for all $k=1,\dots,n$. Then we obtain a contradiction by considering the itineraries $(\dots,x_0,x_1,\dots)$ and $(\dots,z_0,z_1,\dots)$ defined by $x_m=i_k$ and $z_m=j_k$ for $m\equiv k \pmod n$.

(2) $\Leftrightarrow$ (3) By checking its definition, the projection π is a continuous function between compact spaces, thus a homeomorphism if and only if bijective. Hence, being $\tilde{\varphi} = \pi \circ \varphi$, the equivalence (2) $\Leftrightarrow$ (3) is established by considering the following commutative diagram:

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "f"'] & \Sigma \arrow[r, "\pi"] & \widetilde{\Sigma} \\ K \arrow[r, "\varphi"] & \Sigma \arrow[r, "\pi"] \arrow[u, "\sigma"] & \widetilde{\Sigma} \arrow[u, "\sigma"'] \end{tikzcd}$$ $\square$

For a reason to be explained in Section 3.1, call *Markov match* any arc connected component of $W^u(K) \cap R_i$. If $\tilde{\varphi}$ is a conjugacy, Markov matches are characterized by the regrouped itinerary in the same way as they are by the itinerary φ:

Lemma 2.11. *Let the regrouped itinerary $\tilde{\varphi}$ be a conjugacy, and x and y be two points of K such that $\tilde{\varphi}(x) = (\dots, \tilde{x}_0, \tilde{x}_1, \dots)$ and $\tilde{\varphi}(y) = (\dots, \tilde{y}_0, \tilde{y}_1, \dots)$. Then, x and y belong to the same Markov match if and only if $\tilde{x}_n = \tilde{y}_n$ for all $n \geqslant 0$.*

Proof. Let $\varphi(x) = (\dots, x_0, x_1, \dots)$ and $\varphi(y) = (\dots, y_0, y_1, \dots)$ be the (non-regrouped) itineraries of x and y, respectively. It is known that if x and y belong to the same Markov match, then $x_n = y_n$ for all $n \geqslant 0$. Therefore, being $\tilde{\varphi} = \pi \circ \varphi$, it must be $\tilde{x}_n = \tilde{y}_n$ for all $n \geqslant 0$.

The converse also holds. Let $x \in K$ be such that $\varphi(x) = (\dots, x_0, x_1, \dots)$ and $\tilde{\varphi}(x) = (\dots, \tilde{x}_0, \tilde{x}_1, \dots)$. Consider the point $z \in K$ such that $\tilde{\varphi}(z) = (\dots, \tilde{z}_0, \tilde{z}_1, \dots)$, with $\tilde{z}_n = \tilde{x}_n$ for all $n \geqslant 0$. Let $\varphi(z) = (\dots, z_0, z_1, \dots)$. Then, $z_n = x_n$ for all $n \geqslant 0$.

By contradiction, assume that there exists $n_1 \geqslant 0$ such that $z_{n_1} \neq x_{n_1}$. This implies that $z_k \neq x_k$ for all $k \geqslant n_1$. (In fact, if there existed $k_1 > n_1$ such that $z_{k_1} = x_{k_1}$, by applying Lemma 2.10 successively to $k_1, k_1 - 1, \dots, n_1 + 1$ we will obtain that $z_{n_1} = x_{n_1}$ which is absurd.)

Now, as in the proof of Lemma 2.9, let $s \in \{1, \dots, M\}$ be an index appearing infinitely many times in the sequence $\{\tilde{x}_n\}_{n \geqslant n_1} = \{\tilde{z}_n\}_{n \geqslant n_1}$. Since the number of couples (i, j) in $A_s \times A_s$ is finite, there exist x_l, x_{l+m}, z_l and z_{l+m} (all in A_s) such that $x_{l+m} = x_l$ and $z_{l+m} = z_l$. Then the existence of the double cycle $(x_l, z_l) \to (x_{l+1}, z_{l+1}) \to \cdots \to (x_{l+m}, z_{l+m})$ gives the contradiction. $\square$

Remark 2.12. According to Bowen (see [4]), a subset R of the non-wandering set is called a *rectangle* if the local product structure $\mathcal{L}$ defined on $K \times K$ (that is, a continuous map associating to every couple (x, y) in $K \times K$ a point $\mathcal{L}(x, y)$ of the intersection $W^s_{loc}(x) \cap W^u_{loc}(y) \subset K$) is stable with respect to R, that is, $\mathcal{L}(x, y)$ belong to R whenever x and y belong to R. In this case, the point $z = \mathcal{L}(x, y)$ has itinerary $\varphi(z) = (\dots, x_{-1}, x_0 = y_0, y_1, \dots)$, if x and y have itineraries $\varphi(x) = (\dots, x_0, x_1, \dots)$ and $\varphi(y) = (\dots, y_0, y_1, \dots)$, respectively.

The condition “no double cycles” implies that the packages of rectangles can be provided with a product structure. For x and y belonging to rectangles in the same package $\bigcup_{i\in A_{\tilde{x}_0}} R_i$, with regrouped itinerary $\tilde{\varphi}(x) = (\ldots, \tilde{x}_0, \tilde{x}_1, \ldots)$ and $\tilde{\varphi}(y) = (\ldots, \tilde{y}_0, \tilde{y}_1, \ldots)$, the application $\widetilde{\mathcal{L}}(x, y)$ associating to (x, y) the unique point z of $K \cap R_{y_0}$ having $(\ldots, \tilde{x}_{-1}, \tilde{x}_0 = \tilde{y}_0, \tilde{y}_1, \ldots)$ as regrouped itinerary, is well defined and continuous.

(The fact that $\tilde{z}_k = \tilde{y}_k$ for all $k \geqslant 0$ implies, by Lemma 2.11, that z lies in the match I_y of R_{y_0} passing through y, hence the sequence $\{z_k\}_{k\geqslant 0}$ is uniquely determined. By Lemma 2.10, the sequence $\{z_k\}_{k\leqslant 0}$ is uniquely determined by the choice $z_0 = y_0$.)

Thus, $z = \widetilde{\mathcal{L}}(x, y)$ is obtained as the intersection of $I_y \subset W^u_{loc}(y)$ with the segment J_x of $W^s(x) \cap R_{y_0}$ containing all the points $w \in K$ whose itineraries verify: $\widetilde{w}_k = \tilde{y}_k$ for all $k \leqslant 0$ and $w_0 = y_0$.

Last, for x and y belonging to the same rectangle, $\widetilde{\mathcal{L}}(x, y)$ coincides with $\mathcal{L}(x, y)$.

Hence, if the regrouping structure admits no double cycles, the packages $\bigcup_{i\in A_l} R_i$ can be considered as the "rectangles" of a *Markov partition in the sense of Bowen*, even if such packages are not rectangles according to the topological definition (see Section 2.1).

3. The necessary condition

This section is devoted to the proof of the following

Proposition B.1. *Let $h : W^u(K) \to W^u(L)$ be a conjugacy between the unstable manifolds of the saturated systems (K, f) and (L, g). Assume that K and L contain neither hyperbolic attractors, nor hyperbolic repellers. Let σ^u be the unstable geometrical type of $\{R_i\}_{i=1}^N$, a generating Markov partition for (K, f).*

Then there exists a generating Markov partition $\{Q_e\}_{e=1}^E$ for (L, g) provided with a regrouping structure $\{G_i\}_{i=1}^N$ without double cycles and such that the regrouped unstable combinatorial type τ^u of $\{\{Q_e\}_{e\in G_i}\}_{i=1}^N$ is equal to σ^u.

As specified in the introduction, we will fix our attention on the trace of the unstable manifold of K on each single rectangle R_i of the Markov partition for (K, f). We show how to interpretate each image $h(W^u(K) \cap R_i)$ as the trace of the unstable manifold of L on finitely many rectangles $\{Q_e\}_{e\in G_i}$ of a Markov partition for (L, g) (see Fig. 6).

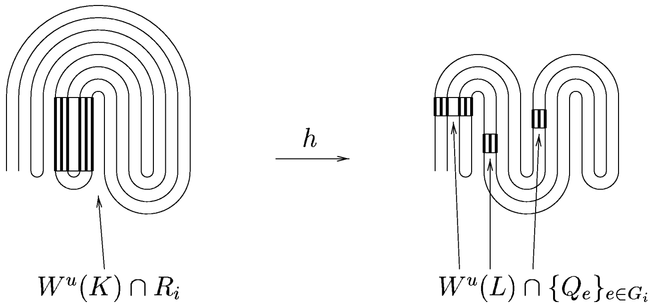

Fig. 6. How to define the regrouped Markov partition for (L, g).

The properties of the regrouping structure will directly follow from the fact that the homeomorphism h is a conjugacy. *Matchboxes* are then the main tool we need to handle.

3.1. Matchboxes and Markov matchboxes

Let $I = [0, 1]$, and J its interior $(0, 1)$. Following [1], we call *matchbox* any topological closed subset M of a lamination $\mathcal{L}$, which is homeomorphic to $C \times I$ where $C \subset I$ is a closed set with empty interior, and whose interior $int(M)$ (with respect to the lamination) is homeomorphic to $C \times J$ via the restricted homeomorphism. We call *match* any of its arc connected components.

We define an *oriented matchbox* as a matchbox provided with a matchbox homeomorphism ϕ, whose matches are oriented via ϕ by the canonical orientation of the intervals I in $C \times I$.

Let $\{R_i\}_{i=1}^N$ be a geometrized Markov partition for (K, f) and $M_i = W^u(K) \cap R_i$ be the trace on $W^u(K)$ of each R_i, $i = 1, \dots, N$. With an abuse of terminology due to the fact that we also admit degenerate rectangles, the sets M_i's will be called *Markov matchboxes*. In the degenerate case, matches are reduced to points but they are still provided with an induced "moral" orientation. Actually, by definition of a rectangle, for a Markov matchbox M_i we can choose a matchbox homeomorphism ϕ such that the orientation on matches induced by ϕ is the same as the one induced by the orientation ω_i on R_i.

So far, matchboxes are defined intrinsically. Nevertheless, the role played by the embedding in the construction of Markov partitions motivates the following

Definition 3.1. An oriented matchbox M contained in a lamination $\mathcal{L}$ lying on a surface S is said to be distinguished if on S there exists a chart (O, ϕ) onto $(-1, 2) \times (-1, 2) \subset \mathbb{R}^2$ such that

- $\phi(O \cap \mathcal{L}) = C \times (-1, 2)$, where C is a closed subset of $[0, 1]$ with empty interior;
- $\phi(M) = C \times [0, 1]$ (or $\phi(M) = C \times \{0\}$ in the degenerate case);
- each match of M is oriented by the canonical orientation of $[0, 1]$ via ϕ.

It is also convenient to introduce the following terminology. Let M be a distinguished matchbox via the chart (O, ϕ). A subset M' of M is called a *transversally smaller matchbox* if:

- M' is a distinguished matchbox via a chart (O', ϕ') such that $O' \subset O$ and $\phi' = \phi|_{O'}$;
- if a point x belongs to M', then the whole match I_x of M passing through x is contained in M', too.

By definition of a rectangle, Markov matchboxes are distinguished matchboxes. In the general case, as a consequence of the compactness of matchboxes and the continuity of the orientation, the following lemma stands:

Lemma 3.2. *Let M be an oriented matchbox and let ω_M be its orientation. Then, there exists a finite family of distinguished disjoint matchboxes $\{M_q\}_{q=1}^Q$ such that $M =$*

$\bigcup_{q=1}^{Q} M_q$ and for which the orientation induced by ω_M is the same as the one induced by the corresponding chart homeomorphism ϕ_q.

Proof. First think just of the embedding and not of the orientation. By definition of a lamination, locally you can always find a trivializing neighborhood. Matches are compact, so you can think of these trivializing neighborhoods as covering matches all along. Any matchbox will therefore be the union of distinguished matchboxes. Because of the transversal compactness of matchboxes, this union is finite.

Call $\{M_p\}_{p=1}^{P}$ such distinguished matchboxes and consider now their induced orientations $\omega_M|_{M_p}$. By continuity, up to considering each matchbox as a finite union of transversally smaller matchboxes $\{M_q\}_{q=1}^{Q}$, we can assume that the image orientation on each matchbox is the same as the one inherited by the corresponding chart homeomorphism ϕ_q. $\square$

3.2. Proof of Proposition B.1

For all $i = 1, \dots, N$, the regrouped rectangles $\{Q_e\}_{e\in S_i}$ of the new Markov partition will be constructed as rectangles exactly covering $h(R_i \cap W^u(K))$.

Let us fix $i \in \{1, \dots, N\}$ and consider the corresponding Markov matchbox M_i with orientation ω_i. By definition, it is clear that the homeomorphic image of a matchbox is again a matchbox. Let $h(M_i)$ be oriented by $h(\omega_i)$. Then, by Lemma 3.2, $h(M_i)$ is a finite union of distinguished oriented disjoint matchboxes $\{\overline{M}_q\}_{q=1}^{Q(i)}$.

The fact that each $\overline{M}_q$ trivially laminates the homeomorphic image of $[0,1]\times[0,1]$ is not sufficient for our purpose. We are interested in matchboxes which trivially laminate a rectangle in our meaning (see Section 2.1).

Lemma 3.3. *The image $h(M_i)$ of the oriented Markov matchbox M_i is a finite union of disjoint oriented matchboxes $\{\overline{M}_p\}_{p=1}^{P(i)}$, each of which is the trace on $W^u(L)$ of an oriented rectangle T_p.*

Proof. Consider $h(M_i) = \{\overline{M}_q\}_{q=1}^{Q(i)}$ as above. Remark that since $\partial_1^s(M_i) = \partial_1^s R_i \cap W^u(K) \subset W^s(x_i)$ for a periodic x_i of K, then by conjugacy $\partial_1^s(\overline{M}_q) \subset W^s(h(x_i)) =: W^s(1,i)$ for all $q = 1, \dots, Q(i)$. Analogously we have that $\partial_2^s(\overline{M}_q) \subset W^s(2,i)$ for all q.

Besides, both $W^s(1,i)$ and $W^s(2,i)$ are isolated by one side. Hence, by continuity, up to choosing transversally smaller matchboxes $\overline{M}_p$'s, we can think that the top (/bottom) endpoints of each $\overline{M}_p$ are the intersection of a certain $\overline{M}_q$ and a segment of $W^s(2,i)$ (/$W^s(1,i)$).

Consider the regions delimited by such segments and by the matches joining their extremities. Fix a region and a match I_0 in the region. Because of the local product structure of the invariant manifolds, there is a distinguished (transversally smaller) matchbox containing I_0, which is the trace on $W^u(L)$ of a rectangle. By using the transversal compactness we are done. $\square$

Repeat the same procedure for all $i = 1, \ldots, N$ and obtain rectangles $\{\{T_p\}_{p=1}^{P(i)}\}_{i=1}^{N}$.

The next step consists in defining two families δ_2^u and δ_2^s of unstable and stable segments, satisfying some good properties (Lemma 3.5) which will make of them the families of the unstable and stable boundaries of the rectangles of a Markov partition.

Remark that the properties established by Lemma 3.5 for δ_2^u are the same as the ones for δ_2^s, if we take into account the different dynamical role played by the two families. However, we want to emphasize that the data at our disposal are not as symmetric as the corresponding statements, which explains the double proof.

Define δ_1^u as the family of the matches constituting $\{\{\partial^u T_p\}_{p=1}^{P(i)}\}_{i=1}^{N}$. Denote δ_1^u by $\{I_a\}_{a\in A}$. Define δ_2^u as the family of matches of $\{\{\overline{M}_p\}_{p=1}^{P(i)}\}_{i=1}^{N}$ for which there exists a positive integer n and an index $a \in A$ such that $g^{-n}(I_a)$ is contained in a match of δ_2^u.

Definition 3.4. A segment $[x, y]$ contained in a stable manifold $W^s(L)$ is called a stable arch if its intersection with the hyperbolic set L consists exactly in its extremal points x and y. In the same way we define an unstable arch.

Let δ_1^s be the family of the (maybe degenerate) stable segments constituting $\{\{\partial^s T_p\}_{p=1}^{P(i)}\}_{i=1}^{N}$. Remove from all segments in δ_1^s the open stable arches having the two extremities in δ_2^u and at least one extremity on $\delta_2^u \setminus \delta_1^u$. Denote by δ_2^s the family of the new (maybe degenerate) segments.

Lemma 3.5. *The families δ_2^u and δ_2^s satisfy the following properties*:

(1) *they are finite*;
(2) *for each match $I \in \delta_2^u$ $(I \in \delta_2^s)$ there exists an open segment $J \subset W^u(L)$ $(J \subset W^s(L))$ such that $J \supset I$ and $J \cap L = I \cap L$*;
(3) *all u-boundary* (*s-boundary*) *periodic points p are covered by an interval of δ_2^u (δ_2^s)*;
(4) *the union of the segments of δ_2^u is invariant under g^{-1}, that is, for all $I \in \delta_2^u$ there exists $I' \in \delta_2^u$ such that $g^{-1}(I) \subset I'$*;
in the same way, the union of the segments of δ_2^s is invariant under g, that is, for all $I \in \delta_2^s$ there exists $I' \in \delta_2^s$ such that $g(I) \subset I'$;
(5) *for all non-periodic point $x \in \delta_2^u \cap L$ $(x \in \delta_2^s \cap L)$ there exists a stable* (*unstable*) *arch starting from x whose other endpoint y belongs to δ_2^u (δ_2^s)*.

Proof. *For the family $\delta_2^u = \{I_b\}_{b\in B}$*: The elements of $\{\{T_p\}_{p=1}^{P(i)}\}_{i=1}^{N}$ are rectangles and they cover L. In particular, any periodic u-boundary point is covered by a rectangle T_p in the family. Being the origin of a stable separatrix not intersecting L, such a point must belong to the unstable boundary $\partial^u T_p$, that is, to a segment of $\delta_1^u \subset \delta_2^u$. Property (3) is then satisfied.

As for property (1), remark first that δ_1^u is finite. Next, for all $a \in A$ and n big enough, $g^{-n}(I_a)$ is contained in the special matches of δ_1^u to which periodic u-boundary points belong. Therefore also $\delta_2^u \setminus \delta_1^u$ is finite.

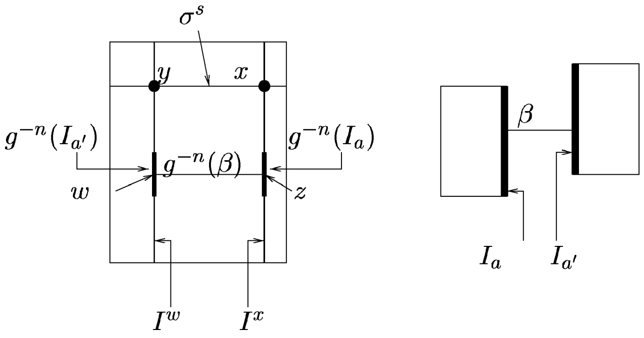

Fig. 7. Stable arches lying on δ_2^u.

Invariance under g^{-1} (property (4)) holds by definition of δ_2^u and by conjugacy. In fact $I_b \supset g^{-n}(I_a)$ for certain n and a, and $g^{-1}(I_b)$ is entirely contained in the same segment $I_{b'}$ containing $g^{-n-1}(I_a)$.

Property (2), according to which the endpoints of matches are isolated in L by one side, holds by conjugacy: all matches of $\{\{\overline{M}_p\}_{p=1}^{P(i)}\}_{i=1}^N$ have the property because they are the conjugate images of matches of Markov matchboxes in $W^u(K)$.

In order to prove the existence of stable arches lying on δ_2^u claimed by property (5), consider a non-periodic point $x \in \delta_2^u \cap L$.

If $x \in g^{-n}(I_a)$ for a certain $n \in \mathbb{N}_0$ and $a \in A$, consider $g^n(x) \in \delta_1^u$. Take a stable arch α starting from $g^n(x)$ lying outside the rectangle T_p to which $g^n(x)$ belongs. Then its other endpoint y belongs to the boundary of a certain rectangle T_q, so to δ_1^u. Therefore $g^{-n}(\alpha)$ is a stable arch between the two points x and $g^{-n}(y)$, both belonging to δ_2^u.

If for any $n \in \mathbb{N}_0$ and for any $a \in A$ the point x does not belong to $g^{-n}(I_a)$, then it belongs to a match $I^x \in \delta_2^u \setminus \delta_1^u$ (see Fig. 7). Consider $a \in A$ and $n \in \mathbb{N}_0$ such that $g^{-n}(I_a) \subset I^x$. Take $z \in g^{-n}(I_a)$ and the arch $g^{-n}(\beta)$ chosen as above. Let $w \neq z$ be its other extremity. By construction there exists $a' \in A$ such that $w \in g^{-n}(I_{a'})$. Consider the match I^w to which w belongs and note that $I^w \in \delta_2^u$. Besides, I^x and I^w are in the same rectangle T_r, so that we can consider the stable segment σ^s crossing T_r and passing through x. Let $y = \sigma^s \cap I^w$. Then the segment contained in σ^s whose endpoints are x and y is the desired arch.

For the family $\delta_2^s = \{I_d\}_{d \in D}$: Property (1) holds: δ_2^s is obtained from a finite family of segments by removing a finite number of open arches.

Property (2) can be verified by using the definition of δ_2^s.

As for property (3), s-boundary periodic points are contained in δ_1^s and no point of L is taken away by the removal.

To prove property (4), consider a segment $I_d \in \delta_2^s$ and its endpoints x and y. Their images $g(x)$ and $g(y)$ are the endpoints of $g(I_d)$ and both belong to the same segment of δ_2^s. In fact, by conjugacy, they lie in the basis of some rectangles in the same group $\{\partial^s T_p\}_{p=1}^{P(i)}$ for a certain $i \in \{1, \ldots, N\}$. Besides, $g(I_d)$ cannot contain in its interior any other point z of δ_2^u (otherwise $g^{-1}(z) \in I_d \cap \delta_2^u$ by the invariance of δ_2^u), that is, $g(I_d)$ is contained in a segment of δ_2^s.

Property (5) holds by conjugacy: for any non-periodic $x \in \delta_2^s \cap L$, $h^{-1}(x)$ belongs to $\partial^s R_i$ for a certain $i \in \{1, \ldots, N\}$. Let α be the unstable arch joining $h^{-1}(x)$ to a point $y \in \partial^s R_j$ for a certain $j \in \{1, \ldots, N\}$. Then $h(\alpha)$ is an unstable arch starting from x and ending up in $h(y) \in L \cap \partial^s T_p$ for a certain p, hence $h(y) \in \delta_2^s$. $\square$

Remark 3.6. No unstable arch with endpoints on δ_2^s is contained in any segment of δ_2^u, as well as no stable arch with endpoints on δ_2^u is contained in any segment of δ_2^s. Therefore, by Theorem 5.3.3 in [3], the two families determine the boundaries of rectangles $\{Q_e\}_{e=1}^E$ of a Markov partition for (L, g).

We end up the proof of Proposition B.1 by showing that conclusions hold for the Markov partition $\{Q_e\}_{e=1}^E$ defined right above.

First remark that by construction the rectangles $\{Q_e\}_{e=1}^E$ are naturally partitioned into the N classes $\{Q_e\}_{e\in G_i}$ in the following way: Q_e and $Q_{\bar{e}}$ belong to the same class G_i if and only if $h^{-1}(Q_e)$ and $h^{-1}(Q_{\bar{e}})$ are contained in the same rectangle R_i.

The Markov partition $\{\{Q_e\}_{e\in G_i}\}_{i=1}^N$ is generating because $\{R_i\}_{i=1}^N$ is. Besides, by construction and by conjugacy, the partition $\{G_i\}_{i=1}^N$ is a regrouping structure. Moreover, by conjugacy, the regrouped unstable combinatorial type τ^u of $\{\{Q_e\}_{e\in G_i}\}_{i=1}^N$ is equal to the unstable combinatorial type σ^u of $\{R_i\}_{i=1}^N$ (just check the definition).

Last, the regrouped incidence matrix of $\{\{Q_e\}_{e\in G_i}\}_{i=1}^N$ is the same as the incidence matrix of $\{R_i\}_{i=1}^N$ because of the way packages are defined. By conjugacy and by Lemma 2.9, the regrouping structure admits no double cycles, and we are done.

4. The sufficient condition

Theorem B is proved if the converse of Proposition B.1 is shown.

Proposition B.2. *Let $W^u(K)$ and $W^u(L)$ be the unstable manifolds of the saturated systems (K, f) and (L, g), respectively. Assume that K and L contain neither hyperbolic attractors, nor hyperbolic repellers. Let $\{R_i\}_{i=1}^N$ be a generating Markov partition for (K, f) and σ^u its unstable combinatorial type. Assume there exists a generating Markov partition $\{Q_p\}_{p=1}^P$ for (L, g) provided with a regrouping structure $\{A_i\}_{i=1}^N$ such that*:

- *the regrouped unstable combinatorial type τ^u of $\{\{Q_p\}_{p\in A_i}\}_{i=1}^N$ equals σ^u;*
- *the regrouping structure has no double cycles.*

Then, there exists a homeomorphism h between $W^u(K)$ and $W^u(L)$ conjugating $f|_{W^u(K)}$ to $g|_{W^u(L)}$.

The remaining subsections correspond to the steps in the definition of such a conjugacy. As it will be specified next, the main ideas are two. First, the combinatorial conditions make it possible to define a conjugacy between the hyperbolic sets K and L; it induces in its turn an order preserving bijection between the oriented Markov matches of R_i for any fixed $i \in \{1, \dots, N\}$, and the oriented Markov matches of the corresponding package $\{Q_p\}_{p \in A_i}$. From the transversal point of view, such a bijection switches our Markov matches, according to a law dictated by the geometrized Markov partitions of the two systems (K, f) and (L, g).

The second idea consists in completing the definition of the conjugacy on *meager ribbons* and *free separatrices* (see Sections 4.2 and 4.3) with the help of the already fixed invariant transversal foliation: we use it to compel a certain transversal rigidity in the neighborhood of free separatrices.

Proposition A can be proved independently by following the same steps and by noticing that it deals with the situation where packages are constituted by only one rectangle.

4.1. On the hyperbolic set

The first step in our construction is to remark that there exists a natural conjugacy defined on the hyperbolic set K onto the hyperbolic set L.

In fact, by Lemma 2.9, the absence of double cycles for the regrouping structure implies that $\tilde{\varphi} : L \to \widetilde{\Sigma} \subset \{1, \dots, N\}^{\mathbb{Z}}$ is a conjugacy between $g^{-1}|_L$ and the subshift $\sigma|_{\widetilde{\Sigma}}$. This gives the right side of the commutative diagram below.

As for the left side, the assumption on the equality between the regrouped unstable combinatorial type τ^u of $\{\{Q_p\}_{p \in A_i}\}_{i=1}^N$ and the unstable combinatorial type σ^u of $\{R_i\}_{i=1}^N$, makes the regrouped incidence matrix of $\{\{Q_p\}_{p \in A_i}\}_{i=1}^N$ equal to the incidence matrix of $\{R_i\}_{i=1}^N$. After calling $\breve{\varphi}$ the itinerary map for the points of K, we have:

$$\begin{tikzcd} K \arrow[r, "\breve{\varphi}"] \arrow[d, "f"'] & \widetilde{\Sigma} \arrow[r, "\tilde{\varphi}^{-1}"] & L \arrow[d, "g"] \\ K \arrow[r, "\breve{\varphi}"] & \widetilde{\Sigma} \arrow[u, "\sigma"] \arrow[r, "\tilde{\varphi}^{-1}"] & L \end{tikzcd}$$

We denote by h_K the conjugacy between the hyperbolic saturated sets K and L defined above by $h_K = \tilde{\varphi}^{-1} \circ \breve{\varphi}$.

Lemma 4.1. *Two points x and y of K belong to the same Markov match $I \in R_i$ if and only if $h_K(x)$ and $h_K(y)$ belong to the same Markov match $J \in Q_p \subset \{Q_p\}_{p \in A_i}$.*

Moreover, h_K preserves their orientations in the following sense. Let I be nondegenerate and oriented by the orientation ω_i of R_i. If $x <_{\omega_i} y$, then $h_K(x) <_{\overline{\omega}_p} h_K(y)$, according to the orientation induced by the orientation $\overline{\omega}_p$ of Q_p.

(*In the case of double s-boundaries, that is, if I is degenerate, make the convention that "moral" orientations are preserved by definition.*)

Proof. The first part is a direct consequence of Lemma 2.11 concerning the characterization of Markov matches via the regrouped itinerary.

The compatibility with respect to the order is due to the fact that the two Markov partitions have the same unstable combinatorial type up to regrouping. Take x and y such that $\breve{\varphi}(x) = (\ldots, x_0, \ldots)$ and $\breve{\varphi}(y) = (\ldots, y_0, \ldots)$, and $x <_{\omega_{x_0}} y$ in $I \in R_{x_0}$. Since they are different and belong to the same match, there exists $M \leqslant 0$ such that $x_l = y_l$ for all $l \geqslant M$, but $x_{M-1} \neq y_{M-1}$.

If $M = 0$, consider the unstable combinatorial type: there exist j_1 and j_2 in $\{1, \ldots, h_{x_0}\}$ such that $\sigma^u(x_0, j_1) = (x_1, \varepsilon_{x_0, j_1})$ and $\sigma^u(x_0, j_2) = (y_1, \varepsilon_{x_0, j_2})$. Besides, $x <_{\omega_{x_0}} y$ if and only if $j_1 < j_2$.

Such a caracterisation is the same when we consider τ^u and the image points $h_K(x)$ and $h_K(y)$. In other words, we know from above that $h_K(x)$ and $h_K(y)$ belong to $J \subset Q_{p_0} \subset \{Q_p\}_{p \in A_{x_0}}$. Let $\hat{j}_1$ and $\hat{j}_2$ in $\{1, \ldots, h_{x_0}\}$ such that $\tau^u(x_0, \hat{j}_1) = (x_1, \varepsilon_{x_0, \hat{j}_1})$ and $\tau^u(x_0, \hat{j}_2) = (y_1, \varepsilon_{x_0, \hat{j}_2})$. Moreover, $h_K(x) <_{\overline{\omega}_{p_0}} h_K(y)$ if and only if $\hat{j}_1 < \hat{j}_2$. Since $\tau^u = \sigma^u$, it is $\hat{j}_1 = j_1$ and $\hat{j}_2 = j_2$. Then, $h_K(x) <_{\overline{\omega}_{p_0}} h_K(y)$ if and only if $j_1 < j_2$, that is, if and only if $x <_{\omega_{x_0}} y$.

If $M < 0$, then $f^M(x)$ and $f^M(y)$ belong to the same match $\hat{I}$ in $f^M(I) \cap R_{x_M}$ containing the points $z \in K$ such that $\breve{\varphi}(z) = (\ldots, z_0, z_1, \ldots)$ with $z_n = x_{n+M}$ for all $n \geqslant 0$. Remark that the relationship between the image orientation $\omega_{\hat{I}} = f^M(\omega_{x_0})|_{\hat{I}}$ and ω_{x_M} can be found via the unstable combinatorial type σ^u. Consider j_1 and j_2 in $\{1, \ldots, h_{x_M}\}$ such that $\sigma^u(x_M, j_1) = (x_{M+1}, \varepsilon_{x_M, j_1})$ and $\sigma^u(x_M, j_2) = (y_{M+1}, \varepsilon_{x_M, j_2})$.

If $\omega_{\hat{I}} = \omega_{x_M}$, we have that $x <_{\omega_{x_0}} y$ if and only if $f^M(x) <_{\omega_{x_M}} f^M(y)$ if and only if $j_1 < j_2$.

In the opposite case, $x <_{\omega_{x_0}} y$ if and only if $f^M(y) <_{\omega_{x_M}} f^M(x)$ if and only if $j_2 < j_1$.

The proof of the lemma can be completed by applying the same procedure as before. $\square$

4.2. *On free separatrices*

Let F_{x_b} be a leaf of $W^u(K)$ containing a point x_b of K, and s_b one of the two connected components of $F_{x_b} \setminus \{x_b\}$. The separatrix s_b is called *free* if it contains no points of K. Separatrices of this type can be characterized as being the ones whose closures with respect to $W^u(K)$ are of the form $\overline{s_b} = s_b \cup \{x_b\}$ [3, Lemma 3.5.1].

It is known that x_b must be a periodic point belonging to $\partial^s R_{i_b}$ for some $i_b \in \{1, \ldots, N\}$ (therefore an endpoint of a match of a Markov matchbox), and that there is only a finite number of such separatrices (see [7] and [3, Section 3.1]). The set of free separatrices $\{s_b\}_{b=1}^S$ is invariant under f. For all $b \in \{1, \ldots, S\}$, let m_b denote the period of s_b. The restriction $f^{m_b}|_{s_b}$ is then conjugate to a strictly monotonic homeomorphism of $\mathbb{R}$ (onto $\mathbb{R}$).

In our notation, if x_b is a double s-boundary, it corresponds to two different indexes b and $\tilde{b}$ ($x_b = x_{\tilde{b}}$ and $s_b \cap s_{\tilde{b}} = x_b$).

In this subsection we want to define a conjugacy $h_{\mathcal{S}}$ on the free separatrices of $W^u(K)$ which is a continuous extension of the conjugacy h_K already defined on the hyperbolic set K. First of all, we will show that h_K induces a bijection between the set of the free separatrices of $W^u(K)$ and the set of the free separatrices of $W^u(L)$, via their extremities (Remark 4.2). Then, we will define a conjugating homeomorphism between the associated separatrices.

Remark 4.2. The point x of K gives rise to a free separatrix s_b of $W^u(K)$ if and only if its image $h_K(x)$ gives rise to a free separatrix $\hat{s}_b$ of $W^u(L)$ of the same period as s_b.

Free separatrices spring up from periodic components of stable boundaries $\partial^s(R_i)$ (see [3, Section 3.1]) in the following situations.

For the case of free separatrices which are not double boundaries assume, in order to fix ideas, that x is a periodic point of period m, that it is s-boundary and that it belongs, say, to $\partial_1^s R_i$, the lower stable boundary of R_i. Then x is the origin of a free separatrix s_b of $W^u(K)$ of period m. Since h_K is a conjugacy preserving matches and their orientations (Lemma 4.1), then $h_K(x)$ has the same properties as x: it is a periodic point of period m, it is s-boundary but not double s-boundary, and it belongs to $\partial_1^s Q_p$ for a certain $p \in A_i$. Then, it is the origin of a free separatrix $\hat{s}_b$ of $W^u(L)$ of period m. By the same proof, the converse holds, too.

As for double s-boundaries, we just have to pay attention to the fact that the point x of period m may give rise to two separatrices of period $2m$ (instead of m), because of the choice of the side (for the degenerate rectangle containing x, the upper and the lower stable boundaries are not distinguished but they split the nearby region into two parts that may be switched by the dynamics). In any case, by using the unstable combinatorial types we can show that for $h_K(x)$ the situation is always the same as the one occurring in x. Because of our convention in Lemma 4.1, that is, $h_K(x)$ preserving "moral" orientations, the correspondence is naturally given.

Let $\{s_b\}_{b=1}^S$ be the set of free separatrices of $W^u(K)$, and $\{\hat{s}_b\}_{b=1}^S$ the corresponding one for $W^u(L)$. We are now ready to define a conjugacy h_{s_b} on s_b for all $b = 1, \ldots, S$ by the classical method of the fundamental intervals. Fix $b \in \{1, \ldots, S\}$. Let w_b be any point of s_b in $W^u(K)$ and $\widehat{w}_b$ be any point of the corresponding $\hat{s}_b$ in $W^u(L)$. Choose an increasing homeomorphism h_b from the fundamental interval $[w_b, f^{m_b}(w_b)]^u$ of s_b with endpoints w_b and $f^{m_b}(w_b)$, onto the fundamental interval $[\widehat{w}_b, g^{m_b}(\widehat{w}_b)]^u$ of $\hat{s}_b$. The homeomorphism h_b can be extended by conjugacy in a unique way to a conjugacy $h_{O(s_b)}$ defined on the orbit $O(s_b) = \{s_b, f(s_b), \ldots, f^{m_b-1}(s_b)\}$ of s_b onto the orbit $\widehat{O}(\hat{s}_b) = \{\hat{s}_b, g(\hat{s}_b), \ldots, g^{m_b-1}(\hat{s}_b)\}$ of $\hat{s}_b$. Repeat the same procedure for the remaining orbits of separatrices. By construction:

Remark 4.3. Let $\mathcal{S}$ stand for $\bigcup_{b=1}^S s_b$ and $\widehat{\mathcal{S}}$ stand for $\bigcup_{b=1}^S \hat{s}_b$. Denote by $h_{\mathcal{S}}$ the map defined on free separatrices by $h_{\mathcal{S}}|_{s_{b_1}} = h_{O(s_b)}$ if the separatrix s_{b_1} belongs to the orbit of s_b. The map $h_{\mathcal{S}}$ is well defined and conjugates the restriction $f|_{\mathcal{S}}$ to the restriction $g|_{\widehat{\mathcal{S}}}$.

Moreover, also the union map $h_K \cup h_{\mathcal{S}}$ is well defined and conjugates the restriction $f|_{K \cup \mathcal{S}}$, to the restriction $g|_{L \cup \widehat{\mathcal{S}}}$.

4.3. *Meager ribbons and regrouped meager ribbons*

When we take free separatrices and Markov matchboxes away from $W^u(K)$, we are left with families of (open) unstable arches (Definition 3.4). We know from [3, Section 3.5], that the closure (with respect to the unstable lamination) of this remaining set is obtained by adding to it its arches endpoints, plus the closures of all free separatrices.

This subsection is devoted to giving the definitions which will be useful in order to handle such arches, and to describe their dynamical role.

First, given a Markov partition $\{R_i\}_{i=1}^N$, it is convenient to regroup these arches in orbits of N families of $(h_i - 1)$ matchboxes. Remember that the horizontal subrectangles $\{H_i^j\}_{j=1}^{h_i-1}$ of the rectangle R_i are numbered by following its vertical orientation ω_i. For each i, consider the $(h_i - 1)$ connected components $\{G_i^j\}_{j=1}^{h_i-1}$ of the closure of $R_i \setminus \{H_i\}_{i=1}^{h_i}$. Make the convention they are also numbered by following ω_i. Their images under f are the connected components of the closure of $f(\bigcup_{i=1}^N R_i) \setminus (\bigcup_{i=1}^N R_i)$. Following [3], we call them *first generation ribbons*. In general, the *kth generation ribbons* are the connected components of the closure of

$$f^k\left(\bigcup_{i=1}^N R_i\right) \Bigg\backslash \left(\bigcup_{l=0}^{k-1} f^l\left(\bigcup_{i=1}^N R_i\right)\right).$$

They turn out to be the images under f of the $(k-1)$th generation ribbons. Besides, ribbons of different generations are disjoint.

We are interested in the traces of $W^u(K)$ on such ribbons.

Definition 4.4. Let $\{R_i\}_{i=1}^N$ be a Markov partition. A first generation meager ribbon γ_i^j ($j \in \{1, \dots, h_i - 1\}$; $i \in \{1, \dots, N\}$) is defined as being the trace of the unstable manifold on the first generation ribbon fG_i^j, i.e., $\gamma_i^j := W^u(K) \cap fG_i^j$. For all $k \in \mathbb{N}$, its $(k-1)$th image $f^{k-1}\gamma_i^j$ $(= W^u(K) \cap f^k G_i^j)$ will be called *kth generation meager ribbon*.

The name *meager* recalls the topology of the transversal sections which are homeomorphic to closed sets of $\mathbb{R}$ with empty interiors. Remark that for all $k \in \mathbb{N}$, kth generation meager ribbons $f^k\gamma_i^j$ are oriented (by $f^{k+1}\omega_i$) matchboxes.

Consider now a regrouping structure $\{A_l\}_{l=1}^M$ for $\{R_i\}_{i=1}^N$. The following definition will turn out to be useful:

Definition 4.5. Let $\{A_l\}_{l=1}^M$ be a regrouping structure for $\{R_i\}_{i=1}^N$. A first generation regrouped meager ribbon Γ_l^j ($l \in \{1, \dots, M\}$; $j = 1, \dots, h_i - 1$ if $i \in A_l$) is the union $\bigcup_{i \in A_l} \gamma_i^j$ of the first generation meager ribbons with superscript j, lying in the rectangles $\{R_i\}_{i \in A_l}$ of the package A_l. Analogously, a kth generation regrouped meager ribbon $g^{k-1}\Gamma_l^j$ ($l \in \{1, \dots, M\}$; $j = 1, \dots, h_i - 1$ if $i \in A_l$) is the union $\bigcup_{i \in A_l} f^{k-1}\gamma_i^j$.

The study of the mutual position of the invariant manifolds (see [3, Sections 3.4–3.6]) allows us to rapidly describe the dynamical behavior of a meager ribbon γ_i^j, that is, the proximity of its orbit $\{f^l\gamma_i^j\}_{l\in\mathbb{N}}$ to the free separatrices $\{s_b\}_{b=1}^S$.

For convenience sake, assume that all free separatrices are fixed. Denote by x_b the origin of the separatrix s_b, and by ∂_b^s the connected component of the stable boundary $\partial^s R_{i_b}$ to which x_b belongs. With respect to the separatrix s_b, the orbit $\{f^l\gamma_i^j\}_{l\in\mathbb{N}}$ can behave

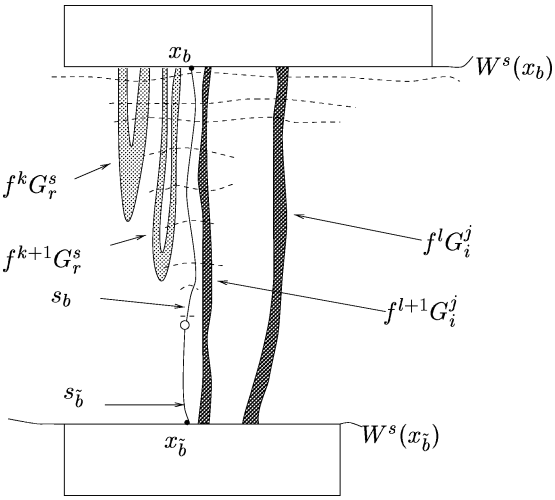

Fig. 8. With respect to the fixed separatrix s_b, it is $\eta_{i,j,b} = 1$ and $\eta_{r,s,b} = 2$.

in three possible ways, which we describe through the position of the endpoints of the matches in $\{f^l\gamma_i^j\}_{l\in\mathbb{N}}$ (see Fig. 8):

(i) s_b does not belong to the adherence of the orbit of the meager ribbon. In particular $f^l\gamma_i^j \cap K \cap \partial_b^s = \emptyset$ for all $l \in \mathbb{N}$;

(ii) s_b belongs to the adherence of the orbit of the meager ribbon, together with another free separatrix $s_{\tilde b}$ (it is the case of the orbit of the meager ribbon γ_i^j in Fig. 8). We describe this situation by the formulation: there exists $l_0(i,j) \in \mathbb{N}$ such that $\emptyset \neq f^l\gamma_i^j \cap \partial_b^s \subsetneq f^l\gamma_i^j \cap K$ for all $l \geqslant l_0(i,j)$;

(iii) s_b is the only free separatrix belonging to the adherence of the orbit of the meager ribbon (the case of the ribbon γ_r^s in Fig. 8). Equivalently: there exists $l_0(i,j) \in \mathbb{N}$ such that $f^l\gamma_i^j \cap K = f^l\gamma_i^j \cap \partial_b^s$ for all $l \geqslant l_0(i,j)$.

In the general case, where s_b is periodic of period m_b, we would have to check separately, for $m_0 = 1, \dots, m_b - 1$, the sequences $\{f^{m_b l + m_0}\gamma_i^j\}_{l\in\mathbb{N}}$.

We will summarize the dynamical behavior of the orbit $\{f^l\gamma_i^j\}_{l\in\mathbb{N}}$ of the meager ribbon γ_i^j by the function

$$\eta_{i,j} : \bigcup_{b=1}^{S} \{b\} \times \{0,1,\dots,m_b-1\} \to \{0,1,2\}$$
$$(b, m_0) \to \eta_{i,j,b,m_0},$$

where $\eta_{i,j,b,m_0} = 0$, 1 or 2 if, with respect to s_b, the orbit $\{f^{m_b l + m_0}\gamma_i^j\}_{l\in\mathbb{N}}$ covers the situation (i), (ii) or (iii), respectively.

By passing, we just say that if the Markov partition $\{R_i\}_{i=1}^N$ is provided with a regrouping structure $\{A_l\}_{l=1}^M$ without double cycles, then the functions $\eta_{i,j}$'s can pass to the quotient: we have that $\eta_{i,j} = \eta_{r,j}$ if i and r belong to the same package $A_{\mathcal{P}(i)}$. We can directly prove it by using the information contained in the regrouped unstable combinatorial type. In our setting, we will obtain the same result (Corollary 4.8) as a corollary of a helpful property: the conjugacy h_K (Section 4.1) induces an orientation preserving bijection between the unstable arches of $W^u(K)$ and the ones of $W^u(L)$ through their endpoints (Lemma 4.6).

4.4. On a neighborhood of free separatrices

Here is the most delicate step in the proof. We want to extend the conjugacy $h_K \cup h_{\mathcal{S}}$ (Remark 4.3) in a neighborhood of free separatrices, while respecting transversal adherences and dynamics. We can do this by means of the invariant stable foliations F^s transversal to $W^u(K)$ and $\widehat{F}^s$ transversal to $W^u(L)$, which were introduced in Proposition 2.3 and which are fixed from now on.

In the following lemmas we construct our main tool: closed invariant neighborhoods $\{\mathcal{W}_b\}_{b=1}^S$ in $W^u(K)$ and closed invariant neighborhoods $\{\widehat{\mathcal{W}}_b\}_{b=1}^S$ in $W^u(L)$ of the free separatrices $\{s_b\}_{b=1}^S$ and $\{\hat{s}_b\}_{b=1}^S$, respectively, satisfying the following properties:

- for any point $x \in \mathcal{W}_b$ there exists a unique unstable segment of arch with endpoints x and $z \in K$, which is entirely contained in $\mathcal{W}_b$, and which we denote by $[z, x]^u$;
- the corresponding point $h_K(z) \in L$ is the endpoint of a unique maximal unstable segment of arch entirely contained in $\mathcal{W}_b$, and which we denote by $[h_K(z), w]^u$;
- if $y \in s_b$ is the projection of x on s_b along the invariant stable foliation F^s, then the leaf of $\widehat{F}^s$ passing through $h_{\mathcal{S}}(y) \in \hat{s}_b$ intersects $[h_K(z), w]^u$ in a unique point $\hat{x}$ belonging to $\widehat{\mathcal{W}}_b$;
- the same properties hold when we interchange F^s with $\widehat{F}^s$.

The points of the couple (z, y) will be considered as the coordinates of the point x, and our construction will make them well defined and continuous. It will be then be possible to associate to $x \in \mathcal{W}_b$ the point $\hat{x} \in \widehat{\mathcal{W}}_b$ corresponding to the couple $(h_K(z), h_{\mathcal{S}}(y))$. This matching will yield a conjugacy from $\mathcal{W}_b$ onto $\widehat{\mathcal{W}}_b$, again because of the care we have taken in the definition of the invariant neighborhoods and the coordinate systems on them.

4.4.1. A bijection between the unstable arches

Here we want to point out that there exists a bijection between the unstable arches lying in ribbons of $W^u(K)$ and the unstable arches lying in ribbons of $W^u(L)$. Such a bijection maps ribbons of $W^u(K)$ to the corresponding regrouped ribbons of $W^u(L)$ of the same generation, preserves the orientations of the arches as well as their dynamical behavior (in the sense of Corollary 4.8).

Take up the same notations as in Proposition B.2. Let $\{\{\gamma_i^j\}_{j=1}^{h_i-1}\}_{i=1}^N$ denote the first generation meager ribbons associated to the Markov partition $\{R_i\}_{i=1}^N$ of $W^u(K)$. Denote by $\{\{\hat{\gamma}_p^q\}_{q=1}^{h_p-1}\}_{p=1}^P$ the first generation meager ribbons corresponding to the Markov partition $\{Q_p\}_{p=1}^P$ of $W^u(L)$, and by $\{\{\widehat{\Gamma}_i^j\}_{j=1}^{h_i-1}\}_{i=1}^N$ the first generation regrouped meager ribbons associated to the regrouping structure $\{A_i\}_{i=1}^N$.

Lemma 4.6. *Let α be an oriented unstable arch of $W^u(K)$, lying in a ribbon $f^k\gamma_i^j$, with endpoints z_1 and z_2 in K, $z_1 < z_2$ along the orientation $f^{k+1}\omega_i$ of α. Then $h_K(z_1)$ and $h_K(z_2)$ are the endpoints in L of an unstable arch $\hat{\alpha}$ of $W^u(L)$ lying in the regrouped* *ribbon $g^k\widehat{\Gamma}_i^j$ and of the same generation as $f^k\gamma_i^j$. Moreover, $h_K(z_1) < h_K(z_2)$ along the orientation $g^{k+1}\overline{\omega}_p$, where $p \in A_i$ is such that $g^{-(k+1)}(h_K(z_1))$ and $g^{-(k+1)}(h_K(z_2))$ belong to Q_p and $\overline{\omega}_p$ is the orientation of Q_p.*

Proof. Recall that h_K preserves matches and orientations (Lemma 4.1). By checking the definition of h_K, the statement is true for meager ribbons of the first generation. By conjugacy, the property holds in the general case. □

Remark 4.7. Because of Lemma 4.1, such a bijection can be extended by conjugacy to all the unstable arches of $W^u(K)$ by iterating negatively the ordered endpoints of the arches in ribbons of the first generation.

Let $\hat{\eta}_{k,j}$ be the functions describing the dynamical behavior of the meager ribbon $\hat{\gamma}_k^j$ in $W^u(L)$.

Corollary 4.8. *Let the meager ribbon γ_k^j of $W^u(L)$ belong to the regrouped meager ribbon Γ_i^j. Then $\hat{\eta}_{k,j} = \eta_{i,j}$ for all $k \in A_i$.*

Proof. The domain is the same because of the correspondence between the sets of the free separatrices of the two systems (K, f) and (L, g) (Remark 4.2). It is then enough to notice that for any arch α in γ_i^j, the values η_{i,j,b,m_0} express the number of endpoints of $f^{lm_b+m_0}\alpha$ lying on ∂_b^s for l big enough. Consider the corresponding arch $\hat{\alpha}$ whose existence is ensured by Lemma 4.6. Its endpoints are the images of the extremities of α via a conjugacy, which leads to the conclusion. □

4.4.2. Construction of the closed invariant neighborhoods $\{\mathcal{W}_b\}_{b=1}^S$ and $\{\widehat{\mathcal{W}}_b\}_{b=1}^S$

Start from considering, for each separatrix s_b, a special neighborhood $\mathcal{N}_b$, the so-called *linearizing collar neighborhood*, whose existence is established in [3, Lemma 4.1.10].

Definition 4.9. Let s_b be a free separatrix of period m_b, springing up from a point x_b of a saturated hyperbolic set. A collar neighborhood of $W^s(x_b)$ is a set $\mathcal{N}_b$ which is homeomorphic via Θ_b to a planar region Ω (see Fig. 9) in $\mathbb{H}^+ = \{(x, y) \in \mathbb{R}^2 \mid y \geqslant 0\}$, externally delimited by the two branches of hyperbola $\{(x, y) \in \mathbb{H}^+ \mid xy = \pm t_0\}$ and containing the horizontal axis $\{(x, y) \in \mathbb{H}^+ \mid y = 0\}$, in such a way that

$$\Theta_b(x_b) = (0, 0), \quad \Theta_b\big(W^s(x_b)\big) = \big\{(x, y) \in \mathbb{H}^+ \mid y = 0\big\} \quad \text{and}$$
$$\Theta_b(s_b) = \big\{(x, y) \in \mathbb{H}^+ \mid x = 0, y > 0\big\}.$$

Moreover, $\mathcal{N}_b$ is called linearizing if it is invariant by f^{m_b} and if the dynamics f^{m_b} on $\mathcal{N}_b$ can be conjugated to the linear hyperbolic application $\mathcal{L}: \Omega \to \Omega$ given by $\mathcal{L}(x, y) = (x/2, 2x)$.

Remark 4.10. It is shown in [3, Lemma 4.1.10], that there exists a family $\{\mathcal{U}_b\}_{b=1}^S$ of linearizing collar neighborhoods of the free separatrices, which are contained in the

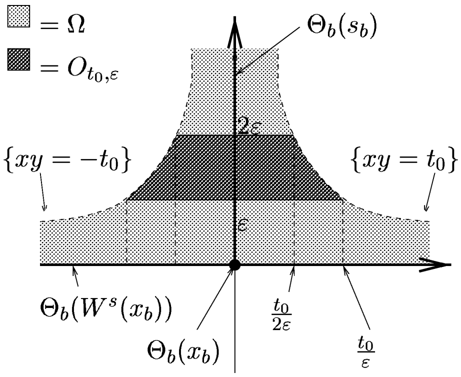

Fig. 9. The planar region Ω and a fundamental domain $O_{t_0,\varepsilon}$.

invariant neighborhood U foliated by F^s, and such that $\mathcal{U}_{b_1}$ and $\mathcal{U}_{b_2}$ are disjoint if and only if $x_{b_1} \neq x_{b_2}$ (if $x_{b_1} = x_{b_2}$, the intersection is given by $W^s(x_{b_1})$).

Fix now b in $\{1, \ldots, S\}$.

Consider in Ω the fundamental domain $O_{t_0,\varepsilon}$ for $t_0 \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}^+$ fixed (see Fig. 9): it is the set of points $(x, y) \in \Omega$ such that $\varepsilon \leqslant y \leqslant 2\varepsilon$. Let $\mathcal{O}_b$ be ${\Theta_b}^{-1}(O_{t_0,\varepsilon})$, that is, the inverse image in $\mathcal{U}_b$ of $O_{t_0,\varepsilon}$.

Denote by c the inverse image by Θ_b of the delimiting curve $\{(x, y) \in \Omega$ such that $y = \varepsilon\}$ or, equivalently, $\{[-t_0/2\varepsilon, t_0/2\varepsilon] \times \{\varepsilon\}\}$. Remark that, by conjugacy, the inverse image of the other delimiting curve $\{[-t_0/\varepsilon, t_0/\varepsilon] \times \{2\varepsilon\}\}$ is the curve $f^{m_b}(c)$.

Moreover, there is no loss of generality in assuming that c (and, by invariance, $f^{m_b}(c)$) is a leaf of the restricted invariant stable foliation $F^s|_{\mathcal{U}_b}$.

Fix b, that is, the free separatrix s_b of period m_b, and let γ_i^j be a ribbon whose orbit contains s_b in its closure. A direct consequence of the λ-lemma and of the meaning of the values η_{i,j,b,m_0}'s is the following Lemma 4.11: starting from a certain k_0 essentially depending on the orbit of the ribbon γ_i^j, the kth iterates $f^{km_b+m_0}\gamma_i^j$ are (with respect to $\mathcal{O}_b$), in one of the two canonical positions represented in Fig. 10.

Lemma 4.11. *Given $b \in \{1, \ldots, S\}$, let $i \in \{1, \ldots, N\}$, $j \in \{1, \ldots, h_i - 1\}$ and $m_0 \in \{0, \ldots, m_b - 1\}$ be such that $\eta_{i,j,b,m_0} \neq 0$. Then there exists $k_0(i, j, b, m_0) \in \mathbb{N}$ such that for all $k \geqslant k_0(i, j, b, m_0)$:*

(1) *the intersection $f^{km_b+m_0}\gamma_i^j \cap \mathcal{O}_b$ is a matchbox whose matches have one endpoint on c an the other one on $f^{m_b}(c)$;*
(2) *for any arch α in $f^{km_b+m_0}\gamma_i^j$, the intersection $\alpha \cap \mathcal{O}_b$ consists in exactly η_{i,j,b,m_0} arc connected components.*

Now, for all ribbons accumulating on s_b, we can consider corresponding integers k_0's as above. For convenience sake, denote by $k_1(b)$ the maximum of such k_0's, that is, the maximum (on i, j and m_0) of the values $k_0(i, j, b, m_0)$ satisfying properties (1) and (2)

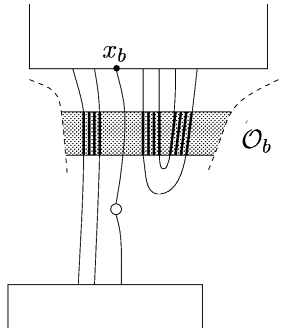

Fig. 10. The canonical position of accumulating ribbons.

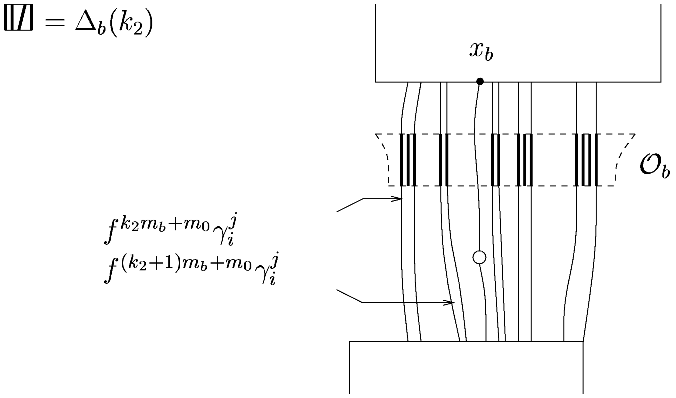

Fig. 11. The construction of $\Delta_b(k_2)$.

in Lemma 4.11. For $k_2 \geqslant k_1(b)$, denote by $\Delta_b(k_2)$ the following union of matchboxes of $W^u(K)$ (see Fig. 11) defined by

$$\Delta_b(k_2) = \bigcup_{k \geqslant k_2(b)} \Big\{ f^{km_b+m_0}\gamma_i^j \cap \mathcal{O}_b \text{ with } i \in \{1, \ldots, N\},\ j \in \{1, \ldots, h_i - 1\} \text{ and } m_0 \in \{0, \ldots, m_b - 1\} \text{ such that } \eta_{i,j,b,m_0} \neq 0\Big\}.$$

We will obtain an f^{m_b}-invariant closed neighborhood of s_b in Lemma 4.13, for which we need the following tool, already proved in [3, Section 3.5].

Lemma 4.12. *Let $\{y_n\}$ be a sequence of points of $W^u(K)$ converging to a point y of a free separatrix s_b. Assume that there exists a sequence $q_n \to +\infty$ such that the points $f^{-q_n}(y_n)$ all belong to an unstable arch α. If z is a point for which there exists a subsequence q_{n_h} of q_n such that $z = \lim_{h\to+\infty} f^{-q_{n_h}}(y_n)$, then z in one of the endpoints of α.*

Proof. The existence of a filtration for the diffeomorphism f on the surface S implies the existence of an open neighborhood $\mathcal{K}$ of K such that:

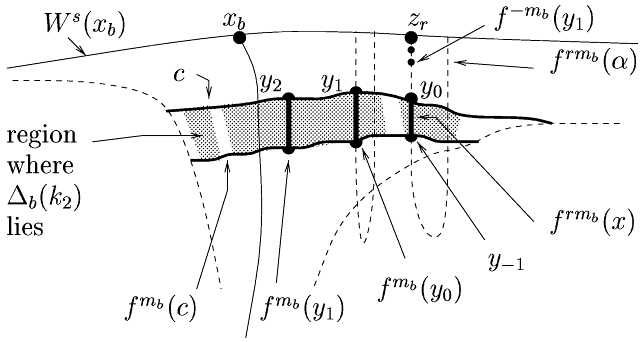

Fig. 12. The arc connected component of $\mathcal{W}_b(k_2)$ containing x.

(1) $f(S \setminus \mathcal{K}) \subset int(S \setminus \mathcal{K})$;

(2) for all $x \notin W^s(K)$, there exists $N \geqslant 0$ such that $f^N(x) \notin \mathcal{K}$.

By (1), up to considering an iterate of $\mathcal{K}$, we can assume that y belongs to $\mathcal{K}$.

Assume by contradiction that $z \notin W^s(K)$. Because of (2), there exists $N \geqslant 0$ such that $f^N(z) \notin \mathcal{K}$, and by continuity of f^N, there also exists a neighborhood $\mathcal{J}$ of z such that $f^N(\mathcal{J}) \cap \mathcal{K} = \emptyset$. By (1), $f^n(\mathcal{J}) \cap \mathcal{K} = \emptyset$ for all $n \geqslant N$. This gives the contradiction: the fact that $z = \lim_{h\to+\infty} f^{-q_{n_h}}(y_n)$, together with $\lim_{n\to+\infty} y_n = y \in \mathcal{K}$, implies that there are infinitely many positive iterates of $\mathcal{J}$ intersecting $\mathcal{K}$.

We are done if we remark that $z \in W^s(K)$ also belongs to $W^u(K)$ (being the limit of points $f^{-q_{n_h}}(y_n)$ all lying on the unstable arch α which is a compact set) and thus, by saturation, to K. $\square$

Lemma 4.13. *For $k_2(b)$ big enough defined above, let $\mathcal{W}_b(k_2)$ be the closure in $W^u(K)$ of $\bigcup_{l\in\mathbb{Z}} f^{lm_b}(\Delta_b(k_2))$. Then, $\mathcal{W}_b(k_2)$ is an f^{m_b}-invariant closed neighborhood of the free separatrix s_b.*

Moreover, let x be a point of $\mathcal{W}_b(k_2)$ and $\alpha = [z_1, z_2]^u$ be the unstable arch (with endpoints z_1 and z_2 in K) containing x. Then, one and only one of the two unstable segments of arch $[z_1, x]^u \subset \alpha$ and $[z_2, x]^u \subset \alpha$ is entirely contained in $\mathcal{W}_b(k_2)$.

Proof. Remark that, by definition, $\mathcal{W}_b(k_2)$ is the union of three sets: the set

$$\bigcup_{l\in\mathbb{Z}} f^{lm_b}\big(\Delta_b(k_2)\big),$$

the set of the points of $W^s(x_b) \cap K$, and the free separatrix s_b. By construction, $\mathcal{W}_b(k_2)$ is a neighborhood in $W^u(K)$ of s_b, it is closed and invariant by f^{m_b}.

Let now x, α, z_1 and z_2 be as in our assumption.

First assume that $x \in \mathcal{W}_b \setminus K$, and show that there exist unique $w \in \mathcal{W}_b \setminus K$ and $z_x \in \{z_1, z_2\}$ such that the unstable segment with endpoints w and z_x is contained in α and contains x.

By definition of $\mathcal{W}_b$, since $x \notin K$, there exists a unique $r \in \mathbb{Z}$ such that $f^{rm_b}(x) \in \Delta_b$ (see Fig. 12). Consider $f^{rm_b}(\alpha)$ and the arc connected component of $f^{rm_b}(\alpha) \cap \mathcal{O}_b$

containing $f^{rm_b}(x)$, which exists by construction (Lemma 4.11). By the same lemma, such a component is an unstable segment with endpoints $y_{-1} \in f^{m_b}(c)$ and $y_0 \in c$. Consider the sequence $\{y_n\}_{n\in\mathbb{N}_0}$, $y_n \in c$, inductively defined by the property: y_n and $f^{m_b}(y_{n-1})$ are the endpoints of a connected component I_n of $f^{(r+n)m_b}(\alpha)$, uniquely determined by x via the starting point y_0.

By construction, the set $\bigcup_{n\in\mathbb{N}_0} f^{-nm_b}(I_n)$ is an unstable half-open segment contained in $f^{rm_b}(\alpha)$, with endpoints y_{-1} and $z_r = \lim_{n\to+\infty} f^{-nm_b}(y_n)$. By Lemma 4.12, the point z_r belongs to K and is one of the endpoints of $f^{rm_b}(\alpha)$. The inverse images $z_x = f^{-rm_b}(z_r)$ and $w = f^{-rm_b}(y_{-1})$ are the endpoints of the maximal connected component of $\alpha \cap \mathcal{W}_b(k_2)$ containing x.

By construction, such a component is strictly contained in α, which gives uniqueness.

If $x \in K$, then it is one of the endpoints of α. The intersection $\alpha \cap \mathcal{W}_b(k_2)$ has a unique connected component containing x, as established above, and we are done. □

So far, we have described how to build a family $\{\mathcal{W}_b(k)\}_{k\geqslant k_2}$ of invariant closed neighborhoods of the separatrix s_b for a given $b \in \{1,\ldots,S\}$. Consider now the corresponding separatrix $\hat{s}_b = h_{\mathcal{S}}(s_b)$ in $W^u(L)$, of same period m_b, and follow the analogous operating procedure in order to define a family of invariant closed neighborhoods $\{\widehat{\mathcal{W}}_b(k)\}_{k\geqslant \hat{k}_2}$ of $\hat{s}_b$:

(I) Let $\{\widehat{\mathcal{U}}_b\}_{b=1}^{S}$ be a family of disjoint linearizing neighborhoods of the free separatrices of $W^u(L)$, covered by the invariant transversal stable foliation $\widehat{F}^s$, already fixed at the beginning of this section.

- For the given $b \in \{1,\ldots,S\}$, let $\widehat{\mathcal{O}}_b$ be the fundamental domain of $\widehat{\mathcal{U}}_b$ delimited by two curves $\hat{c}$ and $g_b^m(\hat{c})$ such that:
  - both $\hat{c}$ and, by invariance, $g_b^m(\hat{c})$, are leaves of the restricted foliation $\widehat{F}^s|_{\widehat{\mathcal{U}}_b}$;
  - the intersection point $\hat{x} = \hat{c} \cap \hat{s}_b \in L$ is the image under $h_{\mathcal{S}}$ of the intersection point $x = c \cap s_b \in K$.
- Remark that for any $p \in \{1,\ldots,P\}$, $q \in \{1,\ldots,\hat{h}_p - 1\}$ and $m_0 \in \{0,\ldots,m_b - 1\}$ such that $\hat{\eta}_{p,q,b,m_0} \neq 0$, there exists $\hat{k}_0(p,q,b,m_0) \in \mathbb{N}$ satisfying properties (1) and (2) of Lemma 4.11 for g and the meager ribbon $\hat{\gamma}_p^q$.

Now, by Corollary 4.8, $\hat{\eta}_{p,j,b,m_0} = \eta_{i,j,b,m_0}$ for all p's in the same package A_i.

We can then state the following lemma for the regrouped meager ribbons:

Lemma 4.14. *Given $b \in \{1,\ldots,S\}$, let $i \in \{1,\ldots,N\}$, $j \in \{1,\ldots,h_{-}1 \mid p \in A_i\}$ and $m_0 \in \{0,\ldots,m_b - 1\}$ such that $\eta_{i,j,b,m_0} \neq 0$. Let $\bar{k}_0(i,j,b,m_0) = \max_{p\in A_i} \hat{k}_0(p,j,b,m_0)$. Then for all $\hat{k} \geqslant \bar{k}_0(i,j,b,m_0)$:*

(1) *the intersection $g^{km_b+m_0}\widehat{\Gamma}_i^j \cap \widehat{\mathcal{O}}_b$ is a matchbox whose matches have one endpoint on $\hat{c}$ an the other one on $g^{m_b}(\hat{c})$;*

(2) *for any arch $\hat{\alpha}$ in $g^{km_b+m_0}\widehat{\Gamma}_i^j$, the intersection $\hat{\alpha} \cap \widehat{\mathcal{O}}_b$ consists in exactly η_{i,j,b,m_0} arc connected components.*

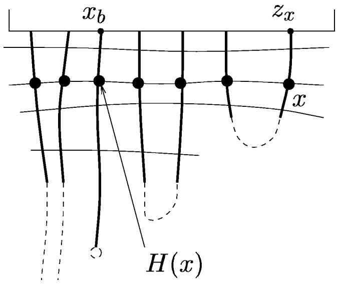

Fig. 13. Holonomy and coordinates.

(II) As done during the construction of Δ_b, denote by $\hat{k}_1(b)$ the maximum (on i, j and m_0) of the values $\bar{k}_0(i, j, b, m_0)$ satisfying the two properties of Lemma 4.14. For all $\hat{k}_2 \geqslant \hat{k}_1(b)$, denote by $\widehat{\Delta}_b(\hat{k}_2)$ the matchbox of $W^u(L)$ defined by

$$\begin{aligned}\widehat{\Delta}_b(\hat{k}_2) = \bigcup_{k \geqslant \hat{k}_2(b)} \Big\{ g^{km_b+m_0} \widehat{\Gamma}_i^j \cap \widehat{\mathcal{O}}_b \text{ with } i \in \{1, \ldots, N\}, \\ j \in \{1, \ldots, h_p - 1 \mid p \in A_i\} \text{ and} \\ m_0 \in \{0, \ldots, m_b - 1\} \text{ such that } \eta_{i,j,b,m_0} \neq 0 \Big\}.\end{aligned}$$

(III) Define $\widehat{\mathcal{W}}_b(\hat{k}_2)$ as the closure in $W^u(L)$ of $\bigcup_{l \in \mathbb{Z}} g^{lm_b}(\widehat{\Delta}_b(\hat{k}_2))$. By Lemma 4.13, $\widehat{\mathcal{W}}_b(\hat{k}_2)$ is a g^{m_b}-invariant closed neighborhood of the free separatrix $\hat{s}_b$ for all $\hat{k}_2 \geqslant \hat{k}_1(b)$.

Given $b \in \{1, \ldots, S\}$, fix $k_3 = \max(k_2, \hat{k}_2)$ defined above, and consider the corresponding neighborhoods $\mathcal{W}_b = \mathcal{W}_b(k_3)$ of s_b and $\widehat{\mathcal{W}}_b = \widehat{\mathcal{W}}_b(\hat{k}_3)$ of $\hat{s}_b$.

For $\tilde{b}$ such that $s_{\tilde{b}} = f^l(s_b)$ is in the same orbit as s_b, let $\mathcal{W}_{\tilde{b}} = f^l(\mathcal{W}_b)$ and $\widehat{\mathcal{W}}_{\tilde{b}} = g^l(\widehat{\mathcal{W}}_b)$.

As far as the remaining orbits of free separatrices are concerned, repeat the same procedure from the beginning.

4.4.3. Coordinates and conjugacy

The choices in the construction of the domains $\{\Delta_b\}_{b=1}^S$ and the definition of the neighborhoods $\{\mathcal{W}_b\}_{b=1}^S$ themselves (Lemma 4.13) make it possible to define on each $\mathcal{W}_b$ the *holonomy* function, which maps any point x of $\mathcal{W}_b$ to the point $H(x)$ of s_b belonging to the same leaf of $F^s|_{\mathcal{W}_b}$ as x (Fig. 13). The holonomy is continuous and onto but not one-to-one: in the picture, all the black points have the same holonomy $H(x)$.

Another consequence of the choices in the construction of $\mathcal{W}_b$ is that we can associate to any point x of $\mathcal{W}_b$ the point z_x of K such that the segment of unstable arch with endpoints x and z_x is entirely contained in $\mathcal{W}_b$ (Lemma 4.13). This function is continuous from $\mathcal{W}_b$ onto $K \cap W^s(x_b)$.

The combination of the two functions we have just defined gives a coordinates system ψ_b on each $\mathcal{W}_b$:

$$\begin{aligned}\psi_b : \mathcal{W}_b &\to \psi(\mathcal{W}_b) \subset s_b \times (K \cap W^s(x_b)) \\ x &\to (H(x), z_x).\end{aligned}$$

In fact, ψ_b turns out to be one-to-one and continuous:

– for $x \notin K$, the sequence $\{x_n\}$ tends to x if and only if for n big enough its elements belong to a rectangular domain $f^r(\Delta_b) \cup H(f^r(\Delta_b))$ (or to two consecutive domains of this type if $x \in \bigcup_{k\in\mathbb{Z}} f^{km_b}c$ where c is defined as in Remark 4.10) on which the convergence clearly is a convergence by coordinates;

– to check continuity on $K \cap W^s(x_b)$, remark that $\lim_{n\to+\infty} x_n = x \in W^s(x_b) \cap K$ if and only if $\lim_{n\to+\infty} H(x_n) = x_b$ and $\lim_{n\to+\infty} z_{x_n} = x$, which is again a convergence by coordinates.

The neighborhood $\mathcal{W}_b$ being compact, ψ_b is a homeomorphism onto its image.

Let ψ be the union map $\bigcup_{b=1}^{S} \psi_b$ defined on $\mathcal{W} = \bigcup_{b=1}^{S} \mathcal{W}_b$: it still is a coordinate function because the $\mathcal{W}_b$'s are either disjoint or the intersection of $\mathcal{W}_b$ and $\mathcal{W}_{\tilde{b}}$ consists in $W^s(x_b) \cap K$, on which ψ_b and $\psi_{\tilde{b}}$ coincide ($\psi_b(x) = (x_b, x) = \psi_{\tilde{b}}(x)$).

Remark 4.15. The invariance of $\mathcal{W}$ and the invariance of the foliation $F^s|_{\mathcal{W}}$ imply that $\psi \circ f = (f \times f) \circ \psi$.

Repeat the same procedure first for $\widehat{\mathcal{W}}_b$ to define $\hat{\psi}_b$ in the analogous way: for $\hat{x} \in \widehat{\mathcal{W}}_b$, it is $\hat{\psi}_b(\hat{x}) = (\widehat{H}(\hat{x}), z_{\hat{x}}) \subset \hat{s}_b \times (L \cap W^s(\hat{x}_b))$. Denote by $\hat{\psi}$ the coordinate function $\bigcup_{b=1}^{S} \psi_b$ defined on $\widehat{\mathcal{W}} = \bigcup_{b=1}^{S} \widehat{\mathcal{W}}_b$. In particular, $\hat{\psi} \circ g = (g \times g) \circ \hat{\psi}$.

An important fact is expressed in the following

Remark 4.16. Let $\bar{h}_{\mathcal{S}} = h_{\mathcal{S}} \cup \ Id_{\{x_b\}_{b=1}^{S}}$ be the map acting as $h_{\mathcal{S}}$ on $\{s_b\}_{b=1}^{S}$, and as the identity on $\{x_b\}_{b=1}^{S}$. The accuracy in the definition of $\widehat{\mathcal{W}}_b$ with respect to $\mathcal{W}_b$ guarantees that the coordinates $\hat{\psi}(\widehat{\mathcal{W}}_b)$ of $\widehat{\mathcal{W}}_b$ are the images under $\bar{h}_{\mathcal{S}} \times h_K$ of the coordinates $\psi(\mathcal{W}_b)$ of $\mathcal{W}_b$: by the "invariance" of the coordinates (Remark 4.15), it is enough to check the property for $\hat{\psi}(\widehat{\Delta}_b)$ and $\psi(\Delta_b)$ for all $b = 1, \ldots, S$, then to complete the proof on $W^s(\hat{x}_b) \cap L$ (for $\hat{x} \in W^s(\hat{x}_b) \cap L$, it is: $\hat{\psi}(\hat{x}) = (\hat{x}_b, \hat{x}) \overset{def}{=} (\bar{h}_{\mathcal{S}}(x_b), h_K(x)) = \psi(x)$).

All these remarks can be summarized in the commutative diagram:

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "\psi"] \arrow[d, "f"'] & \psi(\mathcal{W}) \arrow[r, "\bar{h}_{\mathcal{S}} \times h_K"] \arrow[d, "f\times f"'] & \hat{\psi}(\widehat{\mathcal{W}}) \arrow[r, "\hat{\psi}^{-1}"] \arrow[d, "g\times g"] & \widehat{\mathcal{W}} \arrow[d, "g"] \\
\mathcal{W} \arrow[r, "\psi"] & \psi(\mathcal{W}) \arrow[r, "\bar{h}_{\mathcal{S}} \times h_K"] & \hat{\psi}(\widehat{\mathcal{W}}) \arrow[r, "\hat{\psi}^{-1}"] & \widehat{\mathcal{W}}
\end{tikzcd}$$

It supplies the proof of the following lemma, in which we define a conjugacy between $\mathcal{W}$ and $\widehat{\mathcal{W}}$ via the coordinate functions:

Lemma 4.17. *With the above notations, the application $h_{\mathcal{W}}$ from $\mathcal{W}$ onto $\widehat{\mathcal{W}}$, given by*:

$$h_{\mathcal{W}}(x) = w \quad \textit{if and only if} \quad \hat{\psi}(w) = (\bar{h}_{\mathcal{S}} \times h_K) \circ \psi(x)$$

is well defined and conjugates the restriction $f|_{\mathcal{W}}$ to the restriction $g|_{\widehat{\mathcal{W}}}$.

Moreover, $h_{\mathcal{W}}$ coincides with $h_{\mathcal{S}}$ on the set $\mathcal{S}$ of the free separatrices, and with h_K on the points of $W^s(x_b) \cap \mathcal{W}$ for all $b = 1, \ldots, S$. The union map $h_\star := h_{\mathcal{W}} \cup h_K$ is then well defined and is a conjugacy on its domain $\mathcal{W} \cup K$.

4.5. On the entire unstable manifold

We end up the proof of Proposition B.2.

In Lemma 4.17 we have defined a conjugacy $h_\star$ on $\mathcal{W} \cup K$. Such a domain is invariant, thus so is its complement with respect to $W^u(K)$. The following lemma points out its fundamental domains. Recall that $\{G_i^j\}_{j=1}^{h_i-1}$ denotes the closure of the connected components of $R_i \setminus \{H_i^j\}_{j=1}^{h_i-1}$ for all $i = 1, \ldots, N$ (Section 4.3).

Lemma 4.18. *The set $W^u(K) \setminus (\mathcal{W} \cup K)$ is the union of a finite family of open matchboxes $\{\{T_i^j\}_{j=1}^{h_i-1}\}_{i=1}^N$ together with their iterates $\{f^n(T_i^j)\}_{n\in\mathbb{Z}}$. The closure of each T_i^j is included in the interior of the corresponding G_i^j.*

Proof. The set $W^u(K) \setminus (\mathcal{W} \cup K)$ is the complement of $\mathcal{W}$ in the set of all the unstable arches of $W^u(K)$. By Lemma 4.13, for any unstable arch $\alpha = [z_1, z_2]^u$ in any ribbon, there exist points $x \in \alpha$ and $y \in \alpha$ such that the segments of arch $[z_1, x]^u$ and $[y, z_2]^u$ are covered by $\mathcal{W}$, while the open segment $(x, y)^u$ is in the complement $W^u(K) \setminus \mathcal{W}$. Invariance leads to the conclusion, by considering the element of the orbit $\{f^r((x, y)^u)\}_{n\in\mathbb{Z}}$ belonging to one of the G_i^j's. $\square$

Analogously, $W^u(L) \setminus (\widehat{\mathcal{W}} \cup L)$ is the union of a finite family of open matchboxes $\{\{\widehat{T}_p^q\}_{q=1}^{h_p-1}\}_{p=1}^P$ together with their iterates. The closure of each $\widehat{T}_p^q$ is of course included in the interior of the corresponding $\widehat{G}_p^q$. In particular, remark that $h_{\mathcal{W}}$ induces an order preserving bijection between the matches of any matchbox $T_i^j \subset W^u(K)$ and the matches of the matchboxes $\{\widehat{T}_p^j\}_{p\in A_i} \subset W^u(L)$ in the package A_i, as stated below.

Lemma 4.19. *Let ω_i be the vertical orientation on the rectangle R_i and $\overline{\omega}_p$ the one on Q_p. Let x and y be the endpoints of a match I of $T_i^j \subset R_i$ such that $x <_{\omega_i} y$. Then $h_{\mathcal{W}}(x)$ and $h_{\mathcal{W}}(y)$ are the endpoints of a match $\hat{I}$ of $\widehat{T}_{p_0}^j \subset Q_{p_0}$ such that $p_0 \in A_i$. Moreover, $h_{\mathcal{W}}(x) <_{\overline{\omega}_{p_0}} h_{\mathcal{W}}(y)$.*

The proof is a direct consequence of Remark 4.7 on the correspondence between all the ordered unstable arches, and of the definition of $h_{\mathcal{W}}$ (Lemma 4.13).

Remark 4.20. Denote by Ψ the order preserving bijection between matches defined as in the previous lemma. It is convenient to note that if for $I \subset T_i^j$ we have $\Psi(I) = \hat{I} \subset \widehat{T}_{p_0}^j \subset Q_{p_0}$ for $p_0 \in A_i$, then there exists a distinguished matchbox (Definition 3.1) contained in T_i^j which is entirely mapped by Ψ in $\widehat{T}_{p_0}^j \subset Q_{p_0}$.

Fix the couple (i,j) in $\bigcup_{i=1}^{N}\{i\}\times\{1,\dots,h_i\}$. We want to define a homeomorphism $h_{T_i^j}$ from T_i^j onto $\bigcup_{p\in A_i}\widehat{T}_p^j$ and then transport it by conjugacy on the orbit $O(T_i^j)=\{f^n(T_i^j)\}_{n\in\mathbb{Z}}$ of T_i^j. A convenient way to do it is the following.

Let $P(i)$ be the cardinality of the package A_i. Choose $P(i)$ matches $\{I_p\}_{p\in A_i}$ contained in T_i^j such that $\Psi(I_p)=\hat{I}_p$ is contained in Q_p. Choose $P(i)$ orientation preserving homeomorphisms $\{h_{p,j}\}_{p\in A_i}$ from I_p onto $\hat{I}_p$.

Now, any point $x\in T_i^j$ is the intersection of a match I_x of T_i^j and a leaf F_x of the restricted foliation $F^s|_{R_i}$ (remember that the stable invariant foliation F^s covers the Markov partition, by Proposition 2.3). Then, let p_0 be the unique index in the package A_i such that $\Psi(I_x)\subset Q_{p_0}$. It is natural to associate to x the point $\hat{x}$ which is the intersection of the match $\Psi(I_x)$ with the leaf of the restricted foliation $\widehat{F^s}|_{Q_{p_0}}$ passing through the point $h_{p_0,j}(F_x\cap I_{p_0})$.

Denote by $h_{T_i^j}$ the map we have just defined. By Remark 4.20 it is continuous.

Extend then $h_{T_i^j}$ to the orbit of T_i^j by conjugacy. By definition, the resulting map $h_{O(T_i^j)}$ is a conjugacy on its domain $O(T_i^j)$.

Repeat the procedure for all $j=1,\dots,h_i-1$ and $i=1,\dots,N$.

Consider the union map $h_{\mathcal{T}}=\bigcup_{i,j}h_{O(T_i^j)}$ which is then defined on $W^u(K)\setminus(\mathcal{W}\cup K)$. This union map can be glued to $h_\star$ (defined in Lemma 4.17) thus yielding a conjugacy between $f|_{W^u(K)}$ and $g|_{W^u(L)}$. This finishes the proof of Proposition B.2.

5. Proof of Corollary C

The fact that statement (1) implies statement (2) is trivial. Let us prove the converse by showing that statement (2) in Theorem B holds.

The first step consists in exhibiting the suitable geometrized Markov partitions $\{R_i\}_{i=1}^{N}$ for the system (K,f) and $\{Q_p\}_{p=1}^{P}$ for the system (L,g) which will satisfy such a statement.

Take any Markov partition $\{\widehat{R}_j\}_{j=1}^{R}$ for (K,f). For all $n\in\mathbb{N}$, the connected components of $\bigcup_{j,k=1}^{R}f^{-n}(\widehat{R}_j)\cap\widehat{R}_k$ still form a Markov partition for (K,f). Up to choosing n big enough, we can assume that the new Markov partition $\{R_i\}_{i=1}^{N}$ is such that for all $i=1,\dots,N$ and for all segments I of $f(R_i)\cap W^u(K)$ we have that I is contained in $W_{loc}^u(K)$.

Give now each R_i a vertical orientation ω_i, for $i=1,\dots,N$, and consider the images $\hat{h}(R_i\cap W_{loc}^u(K))$ which are then contained in $W_{loc}^u(L)$. The same is true for the images $\hat{h}(f(R_i)\cap W_{loc}^u(K))$. Now, $\hat{h}$ is a homeomorphism between $W_{loc}^u(K)$ and $W_{loc}^u(L)$ and a conjugacy when restricted to K. So, by the same procedure along Lemmas 3.3 and 3.5, we can obtain here, too, a geometrized Markov partition $\{Q_p\}_{p=1}^{P}$ for the system (L,g) with a natural regrouping structure $\{A_i\}_{i=1}^{N}$ such that:

- for all $p\in A_i$, $i=1,\dots,N$, $\hat{h}(R_i\cap W^u(K))=(\bigcup_{p\in A_i}Q_p)\cap W^u(L)$;
- for all $i=1,\dots,N$, the rectangles $\{Q_p\}_{p\in A_i}$ are oriented by $\hat{h}(\omega_i)$.

Remark 5.1. The only step which is slightly more delicate is the proof of property (5) in Lemma 3.5 for the stable family. The idea is the same, except that we cannot reason directly on the arches α and $\hat{h}(\alpha)$ because $\hat{h}$ is only defined on $W^u_{loc}(K)$; we have then to pass through the arch $f^{-n}\alpha$ (for n big enough) which is contained in $R_i \cap W^u_{loc}(K)$. By considering the arch $g^n(\hat{h}(f^{-n}\alpha))$ we are done.

Because of our choice in the definition of $\{R_i\}_{i=1}^N$, i.e., $\{f(R_i) \cap W^u(K)\}_{i=1}^N \subset W^u_{loc}(K)$, we also have that the regrouped unstable combinatorial type τ^u of $\{\{Q_p\}_{p\in A_i}\}_{i=1}^N$ equals σ^u.

Last, by Lemma 2.9, the regrouping structure admits no double cycles since it is directly defined starting from the conjugacy $\tilde{h}$.

Acknowledgement

I am very much indebted to C. Bonatti for all the mathematical help and encouragement I have received from him. I wish to thank R. Langevin for the stimulating discussions we had. I am grateful to F. Béguin for helpful conversations. I would also acknowledge C. Morales for indicating precious bibliographical references.

References

[1] J.M. Aarts, R.J. Fokkink, On composants of the bucket handle, Fund. Math. 139 (1991) 193–208.
[2] M. Barge, Horseshoe maps and inverse limits, Pacific J. Math. 121 (1986) 29–39.
[3] C. Bonatti, R. Langevin, Difféomorphismes de Smale des surfaces, Prépublication de l'Université de Bourgogne 85 (1996); to appear in Astérisque.
[4] R. Bowen, Markov partitions and minimal sets for Axiom A diffeomorphisms, Amer. J. Math. 92 (1970) 907–918.
[5] M. Hirsch, C. Pugh, M. Shub, Invariant Manifolds, Lecture Notes in Math., Vol. 583, Springer, Berlin, 1970.
[6] E. Jeandenans, Difféomorphismes hyperboliques des surfaces et combinatoire des partitions de Markov, Thèse de l'Université de Bourgogne, Dijon, 1996.
[7] S. Newhouse, J. Palis, Hyperbolic nonwandering sets on two-dimensional manifolds, in: M. Peixoto (Ed.), Dynamical Systems, Academic Press, New York, 1973, pp. 293–301.
[8] S. Smale, Differentiable dynamical systems, Bull. Amer. Math. Soc. 73 (1967) 747–817.
[9] W.T. Watkins, Homeomorphic classification of certain inverse limit spaces with open bonding maps, Pacific J. Math. 103 (1982) 589–601.
[10] R.F. Williams, One-dimensional non-wandering sets, Topology 6 (1967) 473–487.